DERIVATIVE PRICING WITH SYMMETRY ANALYSIS

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Contents

Chapter 1

Introduction

1.1 Economic Motivation

During the past three decades we have witnessed a revolutionary period in the trading of *financial derivative securities* or *contingent claims* in financial markets around the world. A derivative security may be defined as a security whose value depends on the values of other, more basic, underlying variables, which may be the prices of traded securities, prices of commodities or stock indices. There has been a great proliferation in the number and variety of derivative securities and new derivative products are being invented continuously. The construction of the theoretical framework for the pricing of new derivative securities has been one of the major challenges in the field of mathematical finance.

The revolution on derivative securities, both in the stock exchange markets and in academic communities, began in the early 1970's. In 1973, the Chicago Board of Options Exchange started the trading of options in exchanges, although options had been regularly traded by financial institutions in the *over-the-counter* markets previously. In the same year, Black and Scholes [6] and Merton [35] published their seminal papers on the theory of option pricing. Since then the growth of the field of derivative securities has been phenomenal. The Black-Scholes general equilibrium formulation of the option pricing theory is attractive since the final valuation formula deduced from their model is a function of a few observable variables (except one, which is the volatility parameter) so that the accuracy of the model can be ascertained by direct empirical tests with market data. When judged by its ability to explain the empirical data, the option pricing theory is widely acclaimed to be the most successful theory not only in finance, but in all areas of economics. In recognition of their pioneering and fundamental contributions to the pricing theory of derivatives, Scholes and Merton received the 1997 Award of the Nobel Prize in Economics. Unfortunately, Black was unable to receive the same award since he had already passed away then.

Black and Scholes made the following assumptions on the financial market:

- trading takes place continuously in time;
- the riskless interest rate r is known and constant over time;
- the asset pays no dividend;
- there are no transaction costs in buying or selling the asset or the option, and no taxes;
- the assets are perfectly divisible;
- there are no penalties to short selling and the full use of proceeds is permitted;
- the market is complete;
- there are no riskless arbitrage opportunities.

These assumptions have been critically examined by later works in the derivative pricing theory. For example, the interest rate is widely recognized to be fluctuating over time in irregular manner, rather than being constant and deterministic. In the finance community, the Black-Scholes model is still considered to be the most fundamental in derivative pricing theory, although various forms of modification to this basic model have been proposed to accommodate the above shortcomings. However, the last assumption about the absence of arbitrage opportunities remains unchanged and represents a cornerstone of the entire theory.

In their ground-breaking paper [6], Black and Scholes constructed a partial differential equation whose solutions are the valuations of several derivative instruments in terms of the underlying stock's price. Under the standard assumptions listed above, the value $u = u(t, x)$ of the derivative will depend only on the price x of the stock, on time t and on variables that are taken to be known constants. The original Black-Scholes partial differential equation is the following:

$$ru(t,x) = rxu_x(t,x) + \frac{1}{2}\sigma^2 x^2 u_{xx}(t,x) + u_t(t,x) \tag{1.1}$$

where the subscripts are denoting partial derivatives. When the stock provides a continuous dividend stream, we get the *standard Black-Scholes* equation

$$\text{(SBS)} \quad ru(t,x) = (r-q)xu_x(t,x) + \frac{1}{2}\sigma^2 x^2 u_{xx}(t,x) + u_t(t,x) \tag{1.2}$$

where q denotes the constant dividend yield. In particular, the case when the underlying is represented by a future contract can be modelled by equation (1.2) with $q = r$, that is

$$ru(t,x) = \frac{1}{2}\sigma^2 x^2 u_{xx}(t,x) + u_t(t,x) \tag{1.3}$$

The PDE verified by the "delta" (the partial derivative with respect to the stock price x) of a certain financial instrument is

$$u_t(t,x) + \frac{\sigma^2 x^2}{2} u_{xx}(t,x) + (r+\sigma^2)xu_x(t,x) = 0 \tag{1.4}$$

A number of bond pricing models were studied by Merton ([35]), Vasiček ([47]), Cox, Ingersoll and Ross ([16]). Let $p = p(t,r)$ be the bond price, where the interest rate $r = r(t)$ follows a process of the following type

$$dr(t) = f(t,r)dt + g(t,r)dZ(t)$$

where $Z(t)$ denotes a standard Wiener process. If $r(t)$ is characterized by constant parameters $f(t,r) = a,\ g(t,r) = \sigma$, the bond price p satisfies

$$p_t(t,r) + ap_r(t,r) + \frac{\sigma^2}{2}p_{rr}(t,r) - rp(t,r) = 0 \tag{1.5}$$

If $r(t)$ follows an Ornstein-Uhlenbeck process such that $f(t,r) = a(b - r),\ g(t,r) = \sigma$, the equation becomes

$$p_t(t,r) + a(b-r)p_r(t,r) + \frac{\sigma^2}{2}p_{rr}(t,r) - rp(t,r) = 0 \tag{1.6}$$

If $r(t)$ follows a"square root" process $f(t,r) = a(b-r),\ g(t,r) = \sigma\sqrt{r}$, we get the equation

$$p_t(t,r) + a(b-r)p_r(t,r) + \frac{\sigma^2 r}{2} p_{rr}(t,r) - rp(t,r) = 0 \tag{1.7}$$

Finally, the simplest evolutionary linear parabolic second-order partial differential equation is the diffusion equation

$$u_t(t,x) = u_{xx}(t,x) \tag{1.8}$$

that was intensively studied by mathematicians, physicists, or even economists during the last few decades. It can be viewed as a benchmark case in the class of all linear parabolic second order PDE-s, many of them (including several equations derived from financial models) are in fact equivalent to the diffusion equation.

The various equations listed above (1.1-1.8) constitute particular cases of the *generalized Black-Scholes* equation

$$\text{(GBS)} \quad u_t(t,x) = A(t,x)u_{xx}(t,x) + B(t,x)u_x(t,x) + C(t,x)u(t,x) \tag{1.9}$$

where $A(\cdot,\cdot),\ B(\cdot,\cdot),\ C(\cdot,\cdot)$ are indefinitely differentiable functions and $A(\cdot,\cdot)$ is negative almost everywhere. Our aim is to employ *symmetry analysis* to find out properties of (GBS) equation's solution set. Looking at the special case of the (SBS) equation, these properties will be translated into symmetry relationships involving two or more derivative securities' valuations. In particular, pricing formulae for European vanilla options, binary and gap options, choosers, compounds and barrier options will be derived.

1.2 Literature Survey

The pre-history of option theory, leading up to Black-Scholes is described in Briys, Bellalah, Mai and de Varenne [8]. In 1973, Black and Scholes [6], respectively Merton [35], derived the Black-Scholes equation and proved the pricing formulae for European vanilla options. Other ways to derive the Black-Scholes equation are explained in Andreasen, Jensen and Poulson [3]. Several bond pricing models, leading to similar partial differential equations, were studied by Merton [35], Vasiček [47], Cox, Ingersoll and Ross [16].

A significant number of pricing formulae for various derivative instruments were discovered during the last quarter of the century. Black [7] found valuations for options on futures, Garman and Kohlhagen [20] for FX options, Geske [22] for compound options, Rubinstein [41] for simple and complex choosers, Reiner and Rubinstein [40] and Taleb [46] for barrier options, Carr [10] for rainbow options. Zhang discusses in his book [50] many types of exotic options and there is even one book with 1001 formulae for financial derivatives (Haug [24]).

The history of put-call symmetry relationships is even longer:

- **PCP(arity)**. It appears to be originated before 1880 in Castelli [14] and to have been re-discovered almost a century later in Kruizenga [32], [33] and in Stoll [44], [45]. In 1973 Merton [36] states that it is a completely model-free result governing European options values, but not American ones. PCP implies that a put and a call have the same time value.

- **PCE(quivalence)**. It was discovered by Grabbe [23] by studying options on FOREX, where he observed that a put on one currency is a call on the other. Schroder [42] extended PCE to options on many different underlyings. DeTemple [18] showed that PCE governs both European and American options.

- **PCS(ymmetry)** relates an out-of-the-money put on one asset to an out-of-the-money call on the same asset (Bates [5]). It applies to both European and American options and, in contrast to PCE, it allows zero or negative prices for the underlying assets. Carr, Ellis and Gupta generalized in [12] Bates' result to a more general diffusion setting and proved the *binary* put-call symmetry relationship. Carr and Chesney obtained in [11] some results on American put-call symmetry.

- **PCD(uality)** states that the value of a put can be obtained from a formula for valuing a call by negating the stock price, strike price and volatility. It holds for both European and American options (Peskir and Shyriaev [39]).

- **PCR(eversal)** relates the value of a call to the value of a put written on a price process which runs backwards in time. The result is model-dependent and assumes that the usual forward-running stock price process is a jump diffusion, where log-normal volatility is bounded and where the jump part of the return process has independent increments (Andreasen and Carr [2]).

All these results were derived using a wide variety of methods. Some formulae were proved looking at the corresponding boundary conditions satisfied by certain derivative instruments and employing several properties of Black-Scholes equation (e. g., linearity, change of variables, uniqueness of a particular solution), other were constructed using martingale theory (static replication of barrier options with vanilla options in [12]), and sometimes even some basic symmetry properties were mentioned (similarity reduction used to derive the valuation of an exchange option in [48], p. 201).

I will follow a completely different approach in dealing with relationships involving two or more derivative instruments' valuations. My claim is that any such relationship is nothing else than a mathematical expression of certain symmetry admitted by the Black-Scholes equation. If we would list all equation's symmetries and combine them accordingly, then we could derive all the before-mentioned relationships as some well chosen particular symmetries. Consequently, the thesis proposes a *unitary* manner of constructing derivatives' valuations and symmetry relationships among them.

1.3 Mathematical Tools

Around the middle of the nineteenth century Sophus Lie made the profound and far-reaching discovery that all the special techniques designed to solve certain particular types of differential equations (e. g., separable, homogeneous, exact) were, in fact, special cases of a general integration procedure based on the invariance of the differential equation under a continuous group of symmetries. This observation at once unified and significantly extended the available integration techniques, and inspired Lie to devote the remainder of his mathematical career to the development and application of his monumental theory of continuous groups. These groups, now universally known as Lie groups, have had a profound impact on all areas of mathematics, both pure and applied, as well as physics, engineering, economics and other mathematically-based sciences. It is impossible to overestimate the importance of Lie's contribution to modern science and mathematics.

Historically, the applications of Lie groups to differential equations pioneered by Lie and Noether faded into obscurity just as the global, abstract reformulation of differential geometry and Lie group theory championed by E. Cartan gained its ascendency in the mathematical community. The entire subject lay dormant for nearly half a century until G. Birkhoff called attention to the unexploited applications of Lie groups to the differential equations of fluid mechanics. References on their work can be found in Olver's book [37]. Subsequently, Ovsiannikov and his school began a systematic program of successfully applying these methods to a wide range of physically important problems. The last two decades of the twentieth century witnessed a veritable explosion of research activity in this field, both in the applications to concrete physical systems, as well as extensions of the scope and depth of the theory itself. One of the first applications of the afore-mentioned methods to economics and finance is due to Gazizov and Ibragimov [21], who computed all the continuous symmetries of the Black-Scholes and Jacoby-Jones equations.

Roughly speaking, a symmetry group of a (partial) differential equation is a group of mappings which transform solutions to other solutions. In the classical framework of Lie, these groups consist of geometric transformations of the space of independent and dependent variables for the equation, and act on solutions by transforming their graphs. Typical examples are groups of translations and rotations, as well as groups of scaling symmetries, but these certainly do not exhaust the range of possibilities. The great advantage of looking at continuous symmetry groups, as opposed to discrete symmetries such as reflections, is that they can all be found using explicit computational methods. This is not to say that discrete groups are not important in the study of (partial) differential equations, but rather that one must employ quite different methods to find or utilize them. Lie's fundamental discovery was that the complicated nonlinear conditions of equation's invariance under the group transformations could, in the case of a continuous group, be replaced by equivalent, but far simpler, *linear* conditions reflecting a form of "infinitesimal" invariance of the equation under group's generators. In almost every known (partial) differential equation, these infinitesimal symmetry conditions—the so-called determining equations of the Lie symmetry group of the equation—can be explicitly solved in closed form and the most general continuous symmetry group of the equation can be explicitly determined.

1.4 Thesis' Summary

The second chapter presents the mathematical inventory required by the symmetry analysis we are going to perform. In this respect, we will use

Sophus Lie's method described in Olver [37] to get a set of necessary and sufficient conditions for a vector field being an infinitesimal symmetry for (GBS) equation. In the particular case of (SBS) equation we are able to explicitly solve the system of conditions, hence we are going to compute the Lie algebra of the infinitesimal symmetries attached to the standard Black-Scholes equation. Subsequently, the full automorphism group of the Lie algebra will be computed. This automorphism group (more precisely, its corresponding *outer automorphism group*) allows us to derive the complete set of *discrete symmetries*.

The third chapter is concerned with the construction of the standard Black-Scholes equation's symmetries. The continuous ones are obtained from the infinitesimal symmetries by solving a couple of ordinary differential systems attached to them. In order to construct the discrete symmetries, we have to convert every Lie algebra's automorphism into a symmetry via a partial differential system, then to factor out all the continuous symmetries (i. e., the *inner automorphism group*). Finally, we add to the symmetry group the symmetries reflecting the linearity of Black-Scholes equation and we explain the mechanism of constructing new solutions from old ones using continuous and/or discrete symmetries.

In the fourth chapter we give an economic interpretation to (GBS) equation's coefficients and to its symmetries, then we check whether these symmetries are dimensionally correct.

The core of the present thesis is represented by the last chapter. Here we derive valuation formulae for a large number of financial derivatives using symmetry analysis and functional equations. We start with the simplest derivative securities, such as the four types of *binary options*, then we combine them into more complex instruments as *gaps*, *vanilla options*, *choosers*, or *compound options*. At the end of the scale we deal with eight types of weakly path-dependent derivatives, namely with *barrier options*.

1.5 Publications

- Gheorghe Silberberg. "Discrete Symmetries of the Black-Scholes Equation", presented at the *X-th International Conference in MOdern GRoup ANalysis (MOGRAN X), Larnaca, Cyprus, 24-31 October 2004*

- Gheorghe Silberberg. "Computation of Black-Scholes Equation's Full

Symmetry Group", sent to *European Journal of Applied Mathematics*

- Gheorghe Silberberg. "Symmetry Analysis and Vanilla Options", to be sent to *Applied Mathematical Finance*
- Gheorghe Silberberg. "Static Hedging of Barrier Options with Symmetry Analysis", to be sent to *Journal of Finance*
- Gheorghe Silberberg. "Symmetry and Exotic Options", to be sent to *Journal of Finance*

Chapter 2

Mathematical Prerequisites

2.1 Differential Operators and Infinitesimal Symmetries

Let us refer to the general equation (1.9). The basic space representing the independent and dependent variables under consideration is $X \times U \simeq \mathbf{R}^3$, where $X = \{(t, x)\}$ is the space of the independent variables and $U = \{u\}$ is the space of the dependent variable. Then we can define the spaces $U_1 = \{(u_t, u_x)\} \simeq \mathbf{R}^2$, $U_2 = \{(u_{tt}, u_{tx}, u_{xx})\} \simeq \mathbf{R}^3$, $U^{(2)} = U \times U_1 \times U_2 \simeq \mathbf{R}^6$, where a typical point in $U^{(2)}$ looks like

$$u^{(2)} = (u;\ u_t, u_x;\ u_{tt}, u_{tx}, u_{xx}) \tag{2.1}$$

and represents all the derivatives of u of order at most 2 with respect to the variables t and x. $u^{(2)}$ is called the second *prolongation* of function u and it is denoted $pr^{(2)}(u)$. The total space $X \times U^{(2)} \simeq \mathbf{R}^7$ is called the second order *jet space* of the underlying space $X \times U$.
Equation (1.9) can be written as

$$\Delta(t, x; u^{(2)}) = 0$$

where the differential operator $\Delta(\cdot, \cdot; \cdot)$ defined as

$$\Delta(t, x; u^{(2)}) := u_t - A(t, x)u_{xx} - B(t, x)u_x - C(t, x)u \tag{2.2}$$

is a smooth map from the jet space to $\mathbf{R}$. The differential equation itself shows where the given map $\Delta(\cdot, \cdot; \cdot)$ vanishes on $X \times U^{(2)}$, and thus determines a submanifold

$$\mathcal{S}_\Delta := \{(t,x;u^{(2)})|\ \Delta(t,x;u^{(2)}) = 0\} \subseteq X \times U^{(2)}$$

of the total jet space. We can identify the differential equation to its corresponding submanifold. A smooth solution to the given equation is a smooth function $u = u(t,x)$ such that

$$\Delta(t,x;pr^{(2)}u(t,x)) = 0$$

This is just a re-statement of the fact that the partial derivatives of u must satisfy the algebraic constraints imposed by the equation. This condition is equivalent to the statement that the graph of the prolongation $pr^{(2)}u$ must lie entirely within the subvariety $\mathcal{S}_\Delta$.

The Lie *symmetry group* of our equation is a group of transformations $\mathcal{G}$ acting on the space $X \times U$ of the independent and dependent variables, with the property that whenever $u = u(t,x)$ is a solution of the equation then $g \cdot u$ is also a solution for every $g \in \mathcal{G}$. Such a symmetry group acts on the second jet space $X \times U^{(2)}$ as well through its *prolongation* $pr^{(2)}\mathcal{G}$ defined for each element $g \in \mathcal{G}$ as follows:

$$pr^{(2)}g \cdot (t,x;u^{(2)}) := (g \cdot (t,x); (g \cdot u; g \cdot u_t, g \cdot u_x; g \cdot u_{tt}, g \cdot u_{tx}, g \cdot u_{xx}))$$

where $g \cdot u_t$ is the partial derivative of $g \cdot u$ with respect to t and the group action on the other partial derivatives is defined in a similar way. Theorems 2.27 and 2.71 in Olver [37] show that $\mathcal{G}$ is a Lie symmetry group of (GBS) equation if and only if its prolongation leaves the submanifold $\mathcal{S}_\Delta$ invariant. We will look for one-parameter Lie groups that verify the condition above. The general form of such a group is $exp(a\mathbf{v})$, where a is a parameter and $\mathbf{v}$ is a vector field on the jet space. Then the *prolongation* of $\mathbf{v}$, denoted by $pr^{(2)}\mathbf{v}$, is defined to be the infinitesimal generator of the corresponding prolonged one-parameter group $pr^{(2)}[exp(a\mathbf{v})]$. In other words,

$$pr^{(2)}\mathbf{v}|_{(t,x;u^{(2)})} = \frac{d}{da}[exp(a\mathbf{v})](t,x;u^{(2)})\ |_{a=0} \tag{2.3}$$

for any $(t,x;u^{(2)}) \in X \times U^{(2)}$. Theorem 2.8 in Olver [37] allows us to reduce the problem of finding one-parameter Lie symmetry groups of the differential equation to a search for vector fields whose prolongations preserve the submanifold $\mathcal{S}_\Delta$.

2.2 Conditions Satisfied by Continuous Symmetries

2.2.1 Lie's Method

Let

$$\mathbf{v} := \xi^0(t,x,u)\frac{\partial}{\partial t} + \xi^1(t,x,u)\frac{\partial}{\partial x} + \eta(t,x,u)\frac{\partial}{\partial u} \tag{2.4}$$

be a vector field on $X \times U$. Its prolongation is the vector field

$$pr^{(2)}\mathbf{v} := \mathbf{v} + \zeta_0\frac{\partial}{\partial u_t} + \zeta_1\frac{\partial}{\partial u_x} + \zeta_{00}\frac{\partial}{\partial u_{tt}} + \zeta_{01}\frac{\partial}{\partial u_{tx}} + \zeta_{11}\frac{\partial}{\partial u_{xx}} \tag{2.5}$$

where the coefficient functions $\zeta_0, \ldots, \zeta_{11}$ will be determined using the general prolongation formula (2.39) in Olver [37]. The condition that $pr^{(2)}\mathbf{v}$ preserves the submanifold $\mathcal{S}_\Delta$ can be written as

$$pr^{(2)}\mathbf{v}(\Delta) = 0 \qquad \text{whenever} \qquad \Delta = 0 \tag{2.6}$$

Using equations (2.2) and (2.5) we get the *determining equation*

$$\begin{gathered}(-A_t u_{xx} - B_t u_x - C_t u)\xi^0 + (-A_x u_{xx} - B_x u_x - C_x u)\xi^1 - \\ -C\eta + \zeta_0 - B\zeta_1 - A\zeta_{11} = 0\end{gathered} \tag{2.7}$$

The general prolongation formula gives

$$\begin{aligned}\zeta_0 &= D_t(\eta - u_t\xi^0 - u_x\xi^1) + u_{tt}\xi^0 + u_{tx}\xi^1 = \\ &= \eta_t + u_t(\eta_u - \xi^0_t) - u_x\xi^1_t - u_t^2\xi^0_u - u_t u_x\xi^1_u\end{aligned} \tag{2.8}$$

$$\begin{aligned}\zeta_1 &= D_x(\eta - u_t\xi^0 - u_x\xi^1) + u_{tx}\xi^0 + u_{xx}\xi^1 = \\ &= \eta_x - u_t\xi^0_x + u_x(\eta_u - \xi^1_x) - u_t u_x\xi^0_u - u_x^2\xi^1_u\end{aligned} \tag{2.9}$$

$$\begin{gathered}\zeta_{11} = D_x^2(\eta - u_t\xi^0 - u_x\xi^1) + u_{txx}\xi^0 + u_{xxx}\xi^1 = \\ = \eta_{xx} - u_t\xi^0_{xx} + u_x(2\eta_{xu} - \xi^1_{xx}) - 2u_{tx}\xi^0_x + \\ +u_{xx}(\eta_u - 2\xi^1_x) - 2u_t u_x\xi^0_{xu} - u_t u_{xx}\xi^0_u + \\ +u_x^2(\eta_{uu} - 2\xi^1_{xu}) - 2u_x u_{tx}\xi^0_u - 3u_x u_{xx}\xi^1_u - u_t u_x^2\xi^0_{uu} - u_x^3\xi^1_{uu}\end{gathered} \tag{2.10}$$

Knowing that the determining equation (2.7) must hold whenever $u = u(t,x)$ satisfies (1.9), we can substitute for u_t in (2.8), (2.9), (2.10), respectively in (2.7). Then we identify the coefficients of all the monomials (u_x, u_{xx}, u_{tx}, u_x^2, $u_x u_{xx}$ and so on) that appear on both sides of the resulting equation and solve the ordinary differential equations that arise. The solutions represent the infinitesimal generators of the Lie symmetry group we are looking for.

2.2.2 The Determining System

Equating the coefficients of the monomials u_{tx} and $u_x u_{tx}$ to 0, we get

$$\xi^0_x = \xi^0_u = 0$$

hence ξ^0 depends only on time

$$\xi^0 = \xi^0(t) \tag{2.11}$$

Applying the same procedure to the monomials $u_x u_{xx}$ and u_x^2 we get

$$\xi^1_u = \eta_{uu} = 0$$

that is, ξ^1 depends only on the independent variables and η is at most linear with respect to the dependent variable

$$\xi^1 = \xi^1(t,x) \tag{2.12}$$

$$\eta(t,x,u) = \alpha(t,x) + \beta(t,x)u \tag{2.13}$$

The determining equation (2.7) becomes

$$(-A_t u_{xx} - B_t u_x - C_t u)\xi^0 + (-A_x u_{xx} - B_x u_x - C_x u)\xi^1 - \tag{2.14}$$

$$-C(\alpha + \beta u) + \alpha_t + \beta_t u - u_x \xi^1_t + (A u_{xx} + B u_x + C u)(\beta - \xi^0_t) -$$

$$-B[\alpha_x + \beta_x u + u_x(\beta - \xi^1_x)] - A[\alpha_{xx} + \beta_{xx} u + u_x(2\beta_x - \xi^1_{xx}) + u_{xx}(\beta - 2\xi^1_x)] = 0$$

After some re-arrangements we get

$$[A(2\xi^1_x - \xi^0_t) - A_t\xi^0 - A_x\xi^1]u_{xx} + [B(\xi^1_x - \xi^0_t) - B_t\xi^0 - B_x\xi^1 - A(2\beta_x - \xi^1_{xx}) - \xi^1_t]u_x +$$

$$+[\beta_t - A\beta_{xx} - B\beta_x - C_t\xi^0 - C_x\xi^1 - C\xi^0_t]u + (\alpha_t - A\alpha_{xx} - B\alpha_x - C\alpha) = 0$$

which gives the *determining system*

$$\begin{cases} A(2\xi^1_x - \xi^0_t) = A_t\xi^0 + A_x\xi^1 \\ B(\xi^1_x - \xi^0_t) = B_t\xi^0 + B_x\xi^1 + A(2\beta_x - \xi^1_{xx}) + \xi^1_t \\ \beta_t = A\beta_{xx} + B\beta_x + C_t\xi^0 + C_x\xi^1 + C\xi^0_t \\ \alpha_t = A\alpha_{xx} + B\alpha_x + C\alpha \end{cases} \tag{2.15}$$

2.3 Standard Black-Scholes Equation Case

We have

$$A(t,x) = -\frac{\sigma^2 x^2}{2} \qquad B(t,x) = -(r-q)x \qquad C(t,x) = r$$

Suppose that $\mathcal{D} := r - q - \sigma^2/2 \neq 0$. The determining system becomes

$$\begin{cases} x(\xi^1_x - \frac{\xi^0_t}{2}) = \xi^1 \\ -(r-q)x(\xi^1_x - \xi^0_t) = -(r-q)\xi^1 - \sigma^2 x^2(\beta_x - \frac{\xi^1_{xx}}{2}) + \xi^1_t \\ \beta_t = -\frac{\sigma^2 x^2}{2}\beta_{xx} - (r-q)x\beta_x + r\xi^0_t \\ \alpha_t = -\frac{\sigma^2 x^2}{2}\alpha_{xx} - (r-q)x\alpha_x + r\alpha \end{cases} \tag{2.16}$$

Its general solution is computed in Appendix A

$$\begin{cases} \xi^0 &=& C_1 + C_2 t + C_3 t^2 \\ \xi^1 &=& \frac{(C_2 + 2C_3 t)x\log|x|}{2} + (C_4 + C_5 t)x \\ \eta &=& \alpha + \beta u \\ \beta &=& \frac{1}{\sigma^2}\left\{\frac{C_3\log^2|x|}{2} + \frac{[2C_5 - \mathcal{D}(C_2 + 2C_3 t)]\log|x|}{2}\right\} + \omega \\ \omega &=& \left(\frac{\mathcal{D}^2}{2\sigma^2} + r\right)C_3 t^2 + \left[\left(\frac{\mathcal{D}^2}{2\sigma^2} + r\right)C_2 - \frac{C_3}{2} - \frac{\mathcal{D}}{\sigma^2}C_5\right]t + C_6 \end{cases} \tag{2.17}$$

where $C_1, C_2, \ldots, C_6$ are arbitrary constants and α is any solution to the standard Black-Scholes equation.

2.4 The Lie Algebra of Infinitesimal Symmetries

Let $\mathcal{L}_+$ be the set of all the vector fields

$$\mathbf{v} = \xi^0(t,x,u)\frac{\partial}{\partial t} + \xi^1(t,x,u)\frac{\partial}{\partial x} + \eta(t,x,u)\frac{\partial}{\partial u}$$

that satisfy the determining system associated to the standard Black-Scholes equation. According to (2.4) and (2.17), we may write $\mathbf{v} = \mathbf{w} + \mathbf{w_0}$, where

$$\mathbf{w} = (C_1 + C_2 t + C_3 t^2)\frac{\partial}{\partial t} + \left[\frac{(C_2 + 2C_3 t)x \log|x|}{2} + (C_4 + C_5 t)\right]\frac{\partial}{\partial x} +$$

$$+ \left[\frac{1}{\sigma^2}\left\{\frac{C_3 \log^2|x|}{2} + \frac{[2C_5 - \mathcal{D}(C_2 + 2C_3 t)]\log|x|}{2}\right\} + \omega\right] u\frac{\partial}{\partial u}$$

with

$$\omega = \left(\frac{\mathcal{D}^2}{2\sigma^2} + r\right) C_3 t^2 + \left[\left(\frac{\mathcal{D}^2}{2\sigma^2} + r\right) C_2 - \frac{C_3}{2} - \frac{\mathcal{D}}{\sigma^2} C_5\right] t + C_6$$

and

$$\mathbf{w}_0 = \alpha\frac{\partial}{\partial u}$$

Thus $\mathcal{L}_+$ possesses a natural Lie algebra structure and it can be decomposed into a direct sum

$$\mathcal{L}_+ = \mathcal{L} \bigoplus \mathcal{L}_0 = \{\mathbf{w}\} \bigoplus \{\mathbf{w_0}\}$$

where $\mathcal{L}$ consists from all the vector fields in $\mathcal{L}_+$ with $\alpha = 0$ and $\mathcal{L}_0$ is the set of all the vector fields in $\mathcal{L}_+$ with $C_1 = C_2 = \ldots = C_6 = 0$. More precisely,

$$\mathcal{L}_0 = \{\alpha(t, x)\frac{\partial}{\partial u} \mid \alpha \text{ solution to (SBS)}\}$$

is an abelian and infinite dimensional Lie algebra, its subsequent linear space being isomorphic to the space of all solutions to (SBS). We are going to closely examine only the symmetries generated by vector fields contained in $\mathcal{L}$, ignoring those belonging to $\mathcal{L}_0$. There are two reasons for that choice:

- From an algebraic point of view, $\mathcal{L}_0$ is nothing else than a copy of the (SBS) equation's solution set. Therefore an attempt to find out properties of the solution set derived from the structure of $\mathcal{L}_0$ represents a genuine vicious circle.

- The methods that are to be used in the sequel were designed for finite dimensional Lie algebras of infinitesimal symmetries, so we are forced to rely only on information that can be obtained from the study of $\mathcal{L}$.

2.5 The Finite Dimensional Lie Algebra of Infinitesimal Symmetries

2.5.1 Generating Set

Any vector $\mathbf{v}$ belonging to the 6-dimensional Lie algebra $\mathcal{L}$ could be obtained by choosing a coordinate vector

$$\mathcal{C} = \begin{pmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \\ C_5 \\ C_6 \end{pmatrix} \in \mathbf{R}^6$$

A set of six linearly independent coordinate vectors $\{\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_6\}$ corresponds to a generating set $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_6\}$ for $\mathcal{L}$. Let the linearly independent coordinate vectors be

$$\mathcal{C}_1 = \begin{pmatrix} \frac{1}{\sigma^2} \\ 0 \\ 0 \\ \frac{\mathcal{D}}{\sigma^2} \\ 0 \\ \frac{r}{\sigma^2} \end{pmatrix} \qquad \mathcal{C}_2 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \qquad \mathcal{C}_3 = \begin{pmatrix} 0 \\ 2 \\ 0 \\ 0 \\ \mathcal{D} \\ -\frac{1}{2} \end{pmatrix} \tag{2.18}$$

$$\mathcal{C}_4 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \sigma^2 \\ 0 \end{pmatrix} \qquad \mathcal{C}_5 = \begin{pmatrix} 0 \\ 0 \\ \sigma^2 \\ 0 \\ 0 \\ 0 \end{pmatrix} \qquad \mathcal{C}_6 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

The corresponding generating set is represented by the following six operators

$$\begin{cases} \mathbf{v}_1 &= \frac{1}{\sigma^2}\frac{\partial}{\partial t} + \frac{\mathcal{D}}{\sigma^2}x\frac{\partial}{\partial x} + \frac{r}{\sigma^2}u\frac{\partial}{\partial u} \\ \mathbf{v}_2 &= x\frac{\partial}{\partial x} \\ \mathbf{v}_3 &= 2t\frac{\partial}{\partial t} + (\log|x| + \mathcal{D}t)x\frac{\partial}{\partial x} + \left(2rt - \frac{1}{2}\right)u\frac{\partial}{\partial u} \\ \mathbf{v}_4 &= \sigma^2 tx\frac{\partial}{\partial x} + (\log|x| - \mathcal{D}t)u\frac{\partial}{\partial u} \\ \mathbf{v}_5 &= \sigma^2 t^2\frac{\partial}{\partial t} + \sigma^2 tx\log|x|\frac{\partial}{\partial x} + \\ & +\frac{1}{2}[(\log|x| - \mathcal{D}t)^2 + 2\sigma^2 rt^2 - \sigma^2 t]u\frac{\partial}{\partial u} \\ \mathbf{v}_6 &= u\frac{\partial}{\partial u} \end{cases} \tag{2.19}$$

and their commutator table is

	$\mathbf{v}_1$	$\mathbf{v}_2$	$\mathbf{v}_3$	$\mathbf{v}_4$	$\mathbf{v}_5$	$\mathbf{v}_6$
$\mathbf{v}_1$	0	0	$2\mathbf{v}_1$	$\mathbf{v}_2$	$\mathbf{v}_3$	0
$\mathbf{v}_2$	0	0	$\mathbf{v}_2$	$\mathbf{v}_6$	$\mathbf{v}_4$	0
$\mathbf{v}_3$	$-2\mathbf{v}_1$	$-\mathbf{v}_2$	0	$\mathbf{v}_4$	$2\mathbf{v}_5$	0
$\mathbf{v}_4$	$-\mathbf{v}_2$	$-\mathbf{v}_6$	$-\mathbf{v}_4$	0	0	0
$\mathbf{v}_5$	$-\mathbf{v}_3$	$-\mathbf{v}_4$	$-2\mathbf{v}_5$	0	0	0
$\mathbf{v}_6$	0	0	0	0	0	0

The corresponding non-zero structure constants are

$$c^1_{13} = 2 \quad c^2_{14} = 1 \quad c^3_{15} = 1 \quad c^2_{23} = 1$$

$$c^6_{24} = 1 \quad c^4_{25} = 1 \quad c^4_{34} = 1 \quad c^5_{35} = 2$$

and their counterparts

$$c^1_{31} = -2 \quad c^2_{41} = -1 \quad c^3_{51} = -1 \quad c^2_{32} = -1$$

$$c^6_{42} = -1 \quad c^4_{52} = -1 \quad c^4_{43} = -1 \quad c^5_{53} = -2$$

2.5.2 Structure of the Lie Algebra

Let us start by explaining some notations that are to be used in the sequel.

If $\mathbf{v}$ and $\mathbf{w}$ are two elements of a Lie algebra, then $[\mathbf{v}, \mathbf{w}]$ denotes their *commutator* or *Lie bracket*. In the special case when v and w are vector fields defined on a differentiable manifold M and $f : M \to \mathbf{R}$ is a smooth function, then $[\mathbf{v}, \mathbf{w}](\mathbf{f}) := \mathbf{v}(\mathbf{w}(\mathbf{f})) - \mathbf{w}(\mathbf{v}(\mathbf{f}))$.

If $\mathbf{v}_1,\ \mathbf{v}_2,\ \ldots, \mathbf{v}_n$ are arbitrary vector fields, then $\langle \mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n \rangle$ denotes the linear space spanned by them. More precisely,

$$\langle \mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n \rangle := \{\lambda_1 \mathbf{v}_1 + \lambda_2 \mathbf{v}_2 + \ldots + \lambda_n \mathbf{v}_n |\ \lambda_1, \lambda_2, \ldots, \lambda_n \in \mathbf{R}\}$$

Centralizer Structure

It is known that for every operator $\mathbf{v} \in \mathcal{L}$, its associated adjoint action

$$\mathcal{L} \ni \mathbf{w} \mapsto ad_{\mathbf{v}}(\mathbf{w}) = [\mathbf{w}, \mathbf{v}] \in \mathcal{L}$$

is a linear space endomorphism, its kernel being the centralizer $C_{\mathcal{L}}(\mathbf{v})$ of $\mathbf{v}$ and its image being the subspace $[\mathbf{v}, \mathcal{L}]$. As a consequence of the Fundamental Isomorphism Theorem, we have

$$dim(C_{\mathcal{L}}(\mathbf{v})) = codim[\mathbf{v}, \mathcal{L}] \qquad \forall \mathbf{v} \in \mathcal{L}$$

Using the commutator table, we get

$$[\mathbf{v}_1, \mathcal{L}] = \langle \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 \rangle \qquad [\mathbf{v}_2, \mathcal{L}] = [\mathbf{v}_4, \mathcal{L}] = \langle \mathbf{v}_2, \mathbf{v}_4, \mathbf{v}_6 \rangle$$
$$[\mathbf{v}_3, \mathcal{L}] = \langle \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_4, \mathbf{v}_5 \rangle \quad [\mathbf{v}_5, \mathcal{L}] = \langle \mathbf{v}_4, \mathbf{v}_5, \mathbf{v}_6 \rangle \quad [\mathbf{v}_6, \mathcal{L}] = 0$$

It becomes trivial to list all the centralizers of the Lie algebra generators

$$C_{\mathcal{L}}(\mathbf{v}_1) = C_{\mathcal{L}}(\mathbf{v}_2) = \langle \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_6 \rangle \qquad C_{\mathcal{L}}(\mathbf{v}_3) = \langle \mathbf{v}_3, \mathbf{v}_6 \rangle$$
$$C_{\mathcal{L}}(\mathbf{v}_4) = C_{\mathcal{L}}(\mathbf{v}_5) = \langle \mathbf{v}_4, \mathbf{v}_5, \mathbf{v}_6 \rangle \qquad C_{\mathcal{L}}(\mathbf{v}_6) = \mathcal{L}$$

and its center

$$Z(\mathcal{L}) = \bigcap_{i=1}^{6} C_{\mathcal{L}}(\mathbf{v}_i) = \langle \mathbf{v}_6 \rangle$$

Radical

All results and notation in this section are based on Ovsiannikov [38].
The Lie algebra $\mathcal{L}$ can be written as a direct sum

$$\mathcal{L} = \mathcal{R} \bigoplus \mathcal{N}, \qquad \mathcal{R} = \langle \mathbf{v}_2, \mathbf{v}_4, \mathbf{v}_6 \rangle, \quad \mathcal{N} = \langle \mathbf{v}_1, \mathbf{v}_3, \mathbf{v}_5 \rangle \tag{2.20}$$

$\mathcal{R}$ is a solvable ideal of the Lie algebra, its derived series being

$$\mathcal{R} = \langle \mathbf{v}_2, \mathbf{v}_4, \mathbf{v}_6 \rangle \supset \mathcal{R}^{(1)} = \langle \mathbf{v}_6 \rangle \supset \mathcal{R}^{(2)} = \{0\} \tag{2.21}$$

On the other hand, $\mathcal{N}$ is a non-solvable subalgebra, its derived series being stationary. By Lemma 1 p. 177 in Ovsiannikov [38], $\mathcal{N}$ is semi-simple. By the Structural Theorem p. 186 in [38], it is a simple Lie algebra. Summing up, $\mathcal{R}$ is the radical of $\mathcal{L}$ and $\mathcal{N}$ is a corresponding Levi factor ([38], p. 178).

Adjoint Action and Inner Automorphisms

The adjoint action of the one-parameter group generated by $\mathbf{v}_i$ was defined as follows:

$$ad(\mathbf{v}_i) : \mathcal{L} \to \mathcal{L} \quad ad(\mathbf{v}_i)\mathbf{w} = [\mathbf{w}, \mathbf{v}_i] \qquad \forall \mathbf{w} \in \mathcal{L},\ i = 1, 2, \ldots, 6 \qquad (2.22)$$

For every $\lambda \in \mathbf{R}$ and $i \in \{1, 2, \ldots, 6\}$, the exponential map

$$exp(\lambda ad(\mathbf{v}_i)) : \mathcal{L} \to \mathcal{L} \qquad (2.23)$$

is an inner automorphism of the Lie algebra and all inner automorphisms induced by $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_6$ generate the inner automorphism group $Inn(\mathcal{L})$. Their action $\mathbf{v}_{ij}(\lambda) = exp(\lambda ad(\mathbf{v}_i))\mathbf{v}_j$ on the Lie algebra generators can be described as follows: $\mathbf{v}_{ij}(\lambda) = \mathbf{v}_j$ for all $i, j \in \{1, 2, \ldots, 6\}$, with the following exceptions

$$\mathbf{v}_{13}(\lambda) = \mathbf{v}_3 - 2\lambda\mathbf{v}_1, \quad \mathbf{v}_{14}(\lambda) = \mathbf{v}_4 - \lambda\mathbf{v}_2, \quad \mathbf{v}_{15}(\lambda) = \mathbf{v}_5 - \lambda\mathbf{v}_3 + \lambda^2\mathbf{v}_1$$

$$\mathbf{v}_{23}(\lambda) = \mathbf{v}_3 - \lambda\mathbf{v}_2, \quad \mathbf{v}_{24}(\lambda) = \mathbf{v}_4 - \lambda\mathbf{v}_6, \quad \mathbf{v}_{25}(\lambda) = \mathbf{v}_5 - \lambda\mathbf{v}_4 + \frac{\lambda^2}{2}\mathbf{v}_6$$

$$\mathbf{v}_{31}(\lambda) = e^{2\lambda}\mathbf{v}_1, \quad \mathbf{v}_{32}(\lambda) = e^{\lambda}\mathbf{v}_2, \quad \mathbf{v}_{34}(\lambda) = e^{-\lambda}\mathbf{v}_4, \quad \mathbf{v}_{35}(\lambda) = e^{-2\lambda}\mathbf{v}_5$$

$$\mathbf{v}_{41}(\lambda) = \mathbf{v}_1 + \lambda\mathbf{v}_2 + \frac{\lambda^2}{2}\mathbf{v}_6, \quad \mathbf{v}_{42}(\lambda) = \mathbf{v}_2 + \lambda\mathbf{v}_6, \quad \mathbf{v}_{43}(\lambda) = \mathbf{v}_3 + \lambda\mathbf{v}_4$$

$$\mathbf{v}_{51}(\lambda) = \mathbf{v}_1 + \lambda\mathbf{v}_3 + \lambda^2\mathbf{v}_5, \quad \mathbf{v}_{52}(\lambda) = \mathbf{v}_2 + \lambda\mathbf{v}_4, \quad \mathbf{v}_{53}(\lambda) = \mathbf{v}_3 + 2\lambda\mathbf{v}_5$$

Remark 2.1 *Let θ be an inner automorphism of $\mathcal{L}$. Then $\theta(\mathbf{v}_6) = \mathbf{v}_6$. Moreover, the i-th coordinate of $\theta(\mathbf{v}_i)$ is positive for any $i \in \{1, 2, 3, 4, 5\}$.*

The Full Automorphism Group

We find, in Appendix B, all the automorphisms of the Lie algebra $\mathcal{L}$ that are pairwise non-equivalent with respect to the inner automorphism group. There are two types of such automorphisms, namely $\varphi_{\varepsilon\varepsilon'}(\delta)$ and $\psi_{\varepsilon\varepsilon'}(\delta)$, where $\varepsilon, \varepsilon' \in \{-1, 1\}$ and $\delta > 0$. Their description follows.

$\mathbf{v}$	$\mathbf{v}_1$	$\mathbf{v}_2$	$\mathbf{v}_3$	$\mathbf{v}_4$	$\mathbf{v}_5$	$\mathbf{v}_6$
$\varphi_{\varepsilon\varepsilon'}(\delta)(\mathbf{v})$	$\varepsilon\mathbf{v}_1$	$\varepsilon\varepsilon'\delta\mathbf{v}_2$	$\mathbf{v}_3$	$\varepsilon'\delta\mathbf{v}_4$	$\varepsilon\mathbf{v}_5$	$\varepsilon\delta^2\mathbf{v}_6$
$\psi_{\varepsilon\varepsilon'}(\delta)(\mathbf{v})$	$\varepsilon\mathbf{v}_5$	$-\varepsilon\varepsilon'\delta\mathbf{v}_4$	$-\mathbf{v}_3$	$\varepsilon'\delta\mathbf{v}_2$	$\varepsilon\mathbf{v}_1$	$\varepsilon\delta^2\mathbf{v}_6$

The outer automorphism group of the Lie algebra $\mathcal{L}$ is

$$\mathcal{G} = \{\varphi_{\varepsilon\varepsilon'}(\delta), \psi_{\varepsilon\varepsilon'}(\delta)|\ \varepsilon, \varepsilon' \in \{-1, 1\}, \delta > 0\} \tag{2.24}$$

while its full automorphism group is an extension of the inner automorphism group $Inn(\mathcal{L})$ by the outer automorphism group $\mathcal{G}$.

Chapter 3

Symmetries of the Standard Black-Scholes Equation

3.1 Continuous Symmetries

Let

$$\mathbf{v} = \xi^0(t,x,u)\frac{\partial}{\partial t} + \xi^1(t,x,u)\frac{\partial}{\partial x} + \eta(t,x,u)\frac{\partial}{\partial u}$$

be an infinitesimal symmetry and let $\{\Gamma(a) \mid a \in \mathbf{R}\}$ be its corresponding one-parameter Lie symmetry group. Then, according to (2.3), $\Gamma(a)$ transforms the independent and dependent variables $t,\ x,\ u$ into $\hat{t}(a),\ \hat{x}(a),\ \hat{u}(a)$ that satisfy a system of ordinary differential equations

$$\begin{cases} \frac{d\hat{t}}{da} &= \xi^0(\hat{t}(a), \hat{x}(a), \hat{u}(a)) \\ \frac{d\hat{x}}{da} &= \xi^1(\hat{t}(a), \hat{x}(a), \hat{u}(a)) \\ \frac{d\hat{u}}{da} &= \eta(\hat{t}(a), \hat{x}(a), \hat{u}(a)) \end{cases} \tag{3.1}$$

with initial conditions

$$\begin{cases} \hat{t}(0) &= t \\ \hat{x}(0) &= x \\ \hat{u}(0) &= u \end{cases} \tag{3.2}$$

The Lie symmetries that correspond to the six generators of $\mathcal{L}$ are computed in Appendix C.

$$\Gamma_1(a) \begin{cases} \hat{t} &= t + \frac{a}{\sigma^2} \\ \hat{x} &= x \exp\left(\frac{a\mathcal{D}}{\sigma^2}\right) \\ \hat{u} &= u \exp\left(\frac{ar}{\sigma^2}\right) \end{cases} \tag{3.3}$$

$$\Gamma_2(a)\begin{cases}\hat{t} &= t\\ \hat{x} &= x\exp(a)\\ \hat{u} &= u\end{cases} \tag{3.4}$$

$$\Gamma_3(a)\begin{cases}\hat{t} &= t\exp(2a)\\ \hat{x} &= \exp\{\mathcal{D}t[\exp(2a)-\exp(a)]+\log x\exp(a)\}\\ \hat{u} &= u\exp\left\{rt[\exp(2a)-1]-\frac{a}{2}\right\}\end{cases} \tag{3.5}$$

$$\Gamma_4(a)\begin{cases}\hat{t} &= t\\ \hat{x} &= x\exp(a\sigma^2 t)\\ \hat{u} &= u\exp\left[a(\log x-\mathcal{D}t)+\frac{a^2\sigma^2 t}{2}\right]\end{cases} \tag{3.6}$$

$$\Gamma_5(a)\begin{cases}\hat{t} &= \frac{t}{1-a\sigma^2 t}\\ \hat{x} &= \exp\left(\frac{\log x}{1-a\sigma^2 t}\right)\\ \hat{u} &= u\sqrt{|1-a\sigma^2 t|}\exp\left\{\frac{a[(\log x-\mathcal{D}t)^2+2\sigma^2 rt^2]}{2(1-a\sigma^2 t)}\right\}\end{cases} \tag{3.7}$$

$$\Gamma_6(a)\begin{cases}\hat{t} &= t\\ \hat{x} &= x\\ \hat{u} &= u\exp(a)\end{cases} \tag{3.8}$$

3.2 Discrete Symmetries

Discrete symmetries of (partial) differential equations can be used in many ways. They map solutions to (possibly new) solutions. They may be used to create efficient numerical methods for the computation of solutions to boundary-value problems. Discrete and continuous groups of symmetries determine the nature of bifurcations in nonlinear dynamical systems. Equivariant bifurcation theory describes the effects of symmetries, but it may yield misleading results unless *all* symmetries, discrete and continuous, of the dynamical systems are known (see Hydon [29]).

In general, it is straightforward to find all one-parameter Lie groups of symmetries of a given system, using techniques developed by Sophus Lie more than a century ago [37]. Yet, until recently, no method for finding all discrete symmetries was known. The main difficulty is that the determining equations for discrete symmetries typically form a highly-coupled nonlinear system.

A new approach to the problem of finding discrete point symmetries of a partial differential equation has recently been described by Hydon [30]. The technique is based on the observation that every point symmetry yields an automorphism of the Lie algebra of Lie point symmetry generators. This results in a set of auxiliary equations that are satisfied by all point symmetries. These equations can be considerably simplified by factoring out the inner automorphisms of the Lie algebra. After that, they can be solved by standard methods and their solutions are precisely the PDE's discrete symmetries.

Unlike in Hydon [30], the full automorphism group of the Lie algebra was determined using pure algebraic techniques, such as construction of generators' centralizers and Lie algebra's radical. The final results are the description of the outer automorphism group, respectively of the discrete symmetry group associated to Black-Scholes PDE.

Let Γ be a discrete symmetry of the standard Black-Scholes equation that maps the independent and dependent variables (t, x, u) into $(\hat{t}, \hat{x}, \hat{u})$. Then for any generator of the Lie algebra $\mathcal{L}$

$$\mathbf{v}_i = \xi_i^0(t,x,u)\frac{\partial}{\partial t} + \xi_i^1(t,x,u)\frac{\partial}{\partial x} + \eta_i(t,x,u)\frac{\partial}{\partial u} \tag{3.9}$$

we have, according to Hydon [30]

$$\begin{cases} \mathbf{v}_i\hat{t} = \sum_{j=1}^{6} \theta_i^j \xi_j^0(\hat{t},\hat{x},\hat{u}) \\ \mathbf{v}_i\hat{x} = \sum_{j=1}^{6} \theta_i^j \xi_j^1(\hat{t},\hat{x},\hat{u}) \\ \mathbf{v}_i\hat{u} = \sum_{j=1}^{6} \theta_i^j \eta_j(\hat{t},\hat{x},\hat{u}) \end{cases} \tag{3.10}$$

for every $i \in \{1, 2, \ldots, 6\}$. The coefficients θ_i^j are elements of the matrix $\Theta = (\theta_i^j)_{i,j=1,6}$ associated to the automorphism θ that maps the generating system $\{\hat{\mathbf{v}}_i = \Gamma\mathbf{v}_i\Gamma^{-1} |\ i = 1, 6\}$ into $\{\mathbf{v}_i |\ i = 1, 6\}$. If we can solve this system of 18 equations, we would get the components of the symmetry Γ, (i. e., $\hat{t}, \hat{x}, \hat{u}$) as functions of t, x, u. The usual way to do it is to solve the first subsystem of 6 equations (i. e., the equations corresponding to $\hat{t}$) by the method of characteristics, use its solution $\hat{t}(t, x, u)$ to solve the second subsystem, and so on.

All discrete symmetries that are non-equivalent with respect to the continuous ones are computed in Appendix D. There are only four such symmetries, one of them is the identity and the other three are described below.

$$\Gamma_* \begin{cases} \hat{t} &= t \\ \hat{x} &= \exp(2\mathcal{D}t - \log x) \\ \hat{u} &= u \end{cases} \tag{3.11}$$

$$\Gamma_+ \begin{cases} \hat{t} &= -\frac{1}{\sigma^4 t} \\ \hat{x} &= \exp\left[-\frac{1}{\sigma^2 t}\left(\log x - \mathcal{D}t + \frac{\mathcal{D}}{\sigma^2}\right)\right] \\ \hat{u} &= \sigma\sqrt{|t|}\exp\left\{-\frac{1}{2\sigma^2 t}\left[(\log x - \mathcal{D}t)^2 + 2\sigma^2 r t^2 + \frac{2r}{\sigma^2}\right]\right\} u \end{cases} \tag{3.12}$$

$$\Gamma_- \begin{cases} \hat{t} &= -\frac{1}{\sigma^4 t} \\ \hat{x} &= \exp\left[\frac{1}{\sigma^2 t}\left(\log x - \mathcal{D}t - \frac{\mathcal{D}}{\sigma^2}\right)\right] \\ \hat{u} &= \sigma\sqrt{|t|}\exp\left\{-\frac{1}{2\sigma^2 t}\left[(\log x - \mathcal{D}t)^2 + 2\sigma^2 r t^2 + \frac{2r}{\sigma^2}\right]\right\} u \end{cases} \tag{3.13}$$

3.3 The Symmetry Group

Any symmetry of the standard Black-Scholes equation can be written as a product of a continuous symmetry by a discrete one. Since the continuous symmetries are generated by the six one-parameter vector fields listed in section 3.1, and the discrete symmetry group is cyclic of order 4 (see Appendix D, Theorem D.1), we could describe the full symmetry group of the (SBS) equation in terms of its generators.

$$Symm(SBS) = \{\Upsilon \circ \Gamma \mid \Upsilon \text{ Lie symmetry },\ \Gamma \text{ discrete symmetry }\} =$$

$$= \{\prod_{i=1}^{6} \Gamma_i(a_i) \circ \Gamma_+^k \mid a_i \in \mathbf{R},\ k = 0, 1, 2, 3\} \tag{3.14}$$

In practice, the action of a symmetry on a particular solution is equivalent to a chain of actions of some Lie symmetries $\Gamma_i(a_i)$ on that solution preceeded/followed by the action of a discrete symmetry. We could use these symmetries in any order, with any parameters a_i, provided that all groups

$$\{\Gamma_i(a_i) \mid a_i \in \mathbf{R}\}$$

are infinite cyclic normal subgroups of $Symm(SBS)$.

Let us recall that we were forced to ignore certain continuous symmetries when we were studying the Lie algebra $\mathcal{L}$, namely the symmetries corresponding to the vector fields

$$\mathbf{v}_\alpha = \alpha(t, x)\frac{\partial}{\partial u} \qquad \alpha \text{ solution to (SBS)}$$

It is natural to ask ourselves whether $\mathbf{v}_\alpha$ generates new or interesting Lie symmetries. Solving in this case the system (3.1) subject to the initial condition

(3.2), we get

$$\Gamma_\alpha(a)\begin{cases}\hat{t} &= t\\ \hat{x} &= x\\ \hat{u} &= u + a\alpha\end{cases} \tag{3.15}$$

Factoring out the action of $\Gamma_6(\cdot)$ (i. e., the multiplication by a positive constant), the new symmetries we got provide the following well-known results.

Theorem 3.1 *The sum and the difference of two derivative valuations are still derivative valuations.*

We will freely use the conclusion of Theorem 3.1 as an addition to the complete description of the symmetry group in (3.14).

3.4 Construction of New Solutions Using Symmetries

Suppose we are given a solution $u = u(t,x)$ to the standard Black-Scholes equation and a symmetry Γ of the same equation. The action of Γ on u has to define a new solution $\hat{u} = \hat{u}(\hat{t},\hat{x})$ to (SBS) equation which can be constructed in a number of steps.

- Write the explicit action of the symmetry on every variable, dependent or independent.

$$\Gamma\begin{cases}\hat{t} &= \hat{t}(t)\\ \hat{x} &= \hat{x}(t,x)\\ \hat{u} &= \hat{u}(t,x,u)\end{cases}$$

- Reverse the action of the symmetry.

$$\Gamma^{-1}\begin{cases}t &= t(\hat{t})\\ x &= x(\hat{t},\hat{x})\\ u &= u(\hat{t},\hat{x},\hat{u})\end{cases}$$

- Write the new solution using the old variables.

$$\hat{u}(\hat{t},\hat{x}) = \hat{u}(t,x,u(t,x))$$

- Introduce the new variables.

$$\hat{u}(\hat{t},\hat{x}) = \hat{u}(t(\hat{t}),x(\hat{t},\hat{x}),u(t(\hat{t}),x(\hat{t},\hat{x})))$$

Chapter 4

Economic Content of Symmetry Analysis

4.1 Interpretation of Equation's Coefficients

The valuations of many interest rate derivative securities satisfy the (GBS) equation

$$u_t(t,x) = A(t,x)u_{xx}(t,x) + B(t,x)u_x(t,x) + C(t,x)u(t,x) \tag{4.1}$$

when the state variable x follows a diffusion process and t denotes the time. The expression $\sqrt{-2A(t,x)}$ is the state variable's volatility function, while the coefficient $B(t,x)$ is the negative of the state variable's (absolute) risk-neutral drift. The coefficient $C(t,x)$ represents the derivative security's (relative) risk-neutral drift or, equivalently, the cost of carrying the claim once a particular asset is chosen to finance the premium. If this asset is a money market account, then $C(t,x)$ describes the functional relationship between the spot interest rate and the driving state variable in the absence of both default and any continuous cash payouts from the claim (Carr, Lipton and Madan [13]).

In equity derivative models, the state variable x is taken to be the stock price, so that the risk neutral drift $-B(t,x)$ takes the form $[c_x(t,x)-q(t,x)]x$, where $c_x(t,x)$ is the proportional cost of financing positions in the stock and $q(t,x)$ is the dividend yield on the stock. For claims with no intermediate cash flow and in markets with no imperfections or credit risk, the net cost of carry for the derivative security position $C(t,x)$ reduces to the spot interest rate, which is typically assumed to not depend on the stock price. However, correlation between the interest rate and the stock price can be captured in a crude way by allowing this functional dependence $r = C(\cdot, x)$. Furthermore, credit risk and market imperfections can induce a dependence of the derivative's carrying cost on the stock price. For these reasons, it is worthwhile considering the general form (4.1) for both interest rate and equity derivative models. However, we will mainly deal with the standard Black-Scholes setting where

$$A(t,x) = -\frac{\sigma^2 x^2}{2} \qquad B(t,x) = -(r-q)x \qquad C(t,x) = r$$

where the volatility σ, the spot interest rate r and the dividend rate q are taken to be given constants.

There is another parameter that is intensively used in our computations, namely

$$\mathcal{D} := r - q - \frac{\sigma^2}{2}$$

It could be viewed as the drift rate for the logarithm of the asset's price in a risk-neutral world, under the standard assumptions that asset's price follows a geometric Brownian motion and pays continuous dividends at a constant rate q.

4.2 Dimensionality Tests

We could deal with all variables and parameters involved in Black-Scholes equation as if they were abstract quantities, but if we want to add some economic content to the expected results we have to check whether the relationships we use are meaningful in an economic sense. In this respect, we are going to test all of them from the point of view of the *dimensional analysis*. More precisely, we are allowed to add/subtract/compare two quantities only when they could be expressed using the same measurement units.

Let us start with equation's variables. The time t could be measured in seconds, in (trading) days, or even in years. Underlying's price x is usually expressed as a positive multiple of a *numeraire* (USD, EUR, GBP, ...), consequently it could be viewed as a dimensionless variable. The same argument holds for overlying's price $u(t,x)$ as well, but it is not necessary to use the same numeraire for both assets. A typical example is a *quanto*, a derivative contract where the quantum of the payout is applied to a notional principal denominated in some currency other than the natural (domestic) currency in which the underlying asset price is denominated. One of the earliest examples of a quanto to achieve widespread use was the Nikkei 225 average index denominated in USD that has been trading on the Chicago Mercantile Exchange since 1985 (Clewlow and Strickland [15], p. 160).

The constant interest rate r satisfies the money-in-the-bank equation

$$\frac{dM}{M} = r \cdot dt$$

hence its dimensionality is $(\text{time})^{-1}$. Thus its measurement unit is $(\text{seconds})^{-1}$, $(\text{days})^{-1}$ or $(\text{years})^{-1}$ (Wilmott, Howison and Dewynne [49], p. 81). A similar argument holds for the constant dividend rate q too. It is assumed that asset's price x follows a geometric Brownian motion with drift rate μ and volatility σ, that is

$$\frac{dx}{x} = \mu \cdot dt + \sigma \cdot dz$$

where z stands for a one-dimensional standard Brownian motion. Therefore $\dim(z) = (\text{time})^{1/2},\ \dim(\mu) = \dim(\sigma^2) = (\text{time})^{-1}$ and

$$\dim(\mathcal{D}) = \dim\left(r - q - \frac{\sigma^2}{2}\right) = (\text{time})^{-1}$$

It becomes possible to check the correctness of all (SBS) equation's symmetries from a dimensional point of view. Talking about the continuous ones, we should observe that the parameter a is always dimensionless and so is its exponential too. This remark shows that symmetries $\Gamma_1(\cdot),\ \Gamma_2(\cdot),\ \Gamma_3(\cdot), \Gamma_6(\cdot)$ fulfill the dimensionality requirements, that is,

$$\dim(\hat{t}) = \dim(t) \qquad \dim(\hat{x}) = \dim(x) \qquad \dim(\hat{u}) = \dim(u)$$

The same remains true for $\Gamma_4(\cdot)$ and for $\Gamma_5(\cdot)$ as well, since $\log x - \mathcal{D}t$, $\sigma^2 t$, $1 - a\sigma^2 t$ and $\sigma^2 r t^2$ are dimensionless. The symmetries provided by Theorem 3.1 cause no dimensionality problem. Finally, the discrete symmetries are dimensionally correct, since

$$\dim(\sigma^4 t) = \dim(\sigma^4) \cdot \dim(t) = (\text{time})^{-2} \cdot \text{time} = (\text{time})^{-1}$$

and

$$\sigma\sqrt{|t|} \qquad \frac{\mathcal{D}}{\sigma^2} \qquad \frac{r}{\sigma^2}$$

are dimensionless parameters.

4.3 Interpretation of Black-Scholes Equation's Symmetries

Some of the symmetries we constructed in the previous chapter carry a straightforward economic interpretation.

- $\Gamma_\alpha(1)$: $\hat{u}(\hat{t},\hat{u}) = u(\hat{t},\hat{x}) + \alpha(\hat{t},\hat{u})$ shows that the sum of two derivative valuations is a derivative valuation. This remark allows us to establish a lot of parity relationships, starting with the *European put-call parity*

$$(PCP) \qquad CALL = PUT + FUTURES \tag{4.2}$$

- $\Gamma_\alpha(-1)$: $\hat{u}(\hat{t},\hat{x}) = u(\hat{t},\hat{x}) - \alpha(\hat{t},\hat{x})$ shows that the difference of two derivative valuations is a derivative valuation, thus it just "reads" the parity relationships in a different manner. Such relationships are very valuable for hedging purposes. They could also offer arbitrage opportunities or help to synthetically create one of the instruments involved when the other two are available.

- $\Gamma_2(a)$: $\hat{u}(\hat{t},\hat{x}) = u(\hat{t},\exp(-a)\cdot\hat{x})$ changes the numeraire used for the underlying's price. If, for certain derivative instrument, we denominate the underlying asset in other currency than the original one, we get another valid derivative instrument.

- $\Gamma_6(a)$: $\hat{u}(\hat{t},\hat{u}) = \exp(a)\cdot u(\hat{t},\hat{x})$ shows that a similar change of numeraire could be done for the overlying itself.

- $\Gamma_1(\cdot)$ is related to the time translation. Let us define a composite symmetry

$$\tilde{\Gamma}_1(a) := \Gamma_6(a'') \circ \Gamma_2(a') \circ \Gamma_1(a)$$

  where

$$a' = -\frac{a\mathcal{D}}{\sigma^2} \qquad a'' = -\frac{ar}{\sigma^2}$$

  Then the complete action of $\tilde{\Gamma}_1(\cdot)$ on a derivative valuation $u(\cdot,\cdot)$ can be described as follows.

$$\tilde{\Gamma}_1(a)[u(\cdot,\cdot)] = \hat{u}(\cdot,\cdot) \qquad \hat{u}(\hat{t},\hat{x}) = u\left(\hat{t} - \frac{a}{\sigma^2},\hat{x}\right) \tag{4.3}$$

  It only shows that a change of the time origin preserves the property of being a derivative pricing formula.

Chapter 5

Derivative Securities' Valuation

5.1 Binary Options

The least complex financial derivatives are the so-called *binary options*, represented by the *cash-or-nothing*, respectively by the *asset-or-nothing* calls and puts.

The cash-or-nothing call (CONC) pays at expiry date T a fixed amount of money K if underlying's price at that time is larger than the strike price K, otherwise it pays nothing. In other words, its final payoff could be expressed as

$$CONC(T,x) = \begin{cases} K & \text{if} \quad x > K \\ 0 & \text{if} \quad x < K \end{cases} \tag{5.1}$$

Similarly, the final payoff of a cash-or-nothing put (CONP) is

$$CONP(T,x) = \begin{cases} 0 & \text{if} \quad x > K \\ K & \text{if} \quad x < K \end{cases} \tag{5.2}$$

Let us act on the boundary condition (5.1) with a symmetry

$$\Gamma := \Gamma_2(a) \circ \Gamma_* \qquad a = \log K^2 - 2\mathcal{D}T$$

which can be described as follows

$$\Gamma \begin{cases} \hat{t} &= t \\ \hat{x} &= \frac{K^2}{x} \exp[-2\mathcal{D}(T-t)] \\ \hat{u} &= u \end{cases}$$

We have

$$\Gamma[CONC(T,x)] = \begin{cases} K & \text{if } \frac{K^2}{x} > K \\ 0 & \text{if } \frac{K^2}{x} < K \end{cases} = CONP(T,x)$$

We know that $\Gamma[CONC(t,x)]$ has to be the valuation of certain financial derivative that receives the same final payoff as $CONP(t,x)$. Therefore these two derivatives must coincide, hence we proved the following *binary put-call symmetry* relationship

$$CONP(t,x) = CONC\left(t, \frac{K^2}{x}\exp[-2\mathcal{D}(T-t)]\right) \tag{5.3}$$

On the other hand, the sum of the final payoffs for the binary call and put, $CONC(T,x)+CONP(T,x) = K$, is identical to the payoff of a riskless bond with the same expiry date and the same strike price. Since the sum of two derivatives' valuations should be a derivative valuation, we could extend this relationship back in time, getting the *binary put-call parity* relationship

$$CONC(t,x) + CONP(t,x) = K\exp[-r(T-t)] \tag{5.4}$$

Combining equations (5.3) and (5.4) we get a functional equation for the valuation of a cash-or-nothing call

$$CONC(t,x) + CONC\left(t, \frac{K^2}{x}\exp[-2\mathcal{D}(T-t)]\right) = K\exp[-r(T-t)] \tag{5.5}$$

which we can solve, finally getting the well-known expressions (Wilmott [48], p. 175)

$$CONC(t,x) = K\exp[-r(T-t)]N(d^-) \tag{5.6}$$
$$CONP(t,x) = K\exp[-r(T-t)]N(-d^-)$$

where $N(\cdot)$ stands for the cumulative normal distribution function and

$$d^- := \frac{\log(x/K) + \mathcal{D}(T-t)}{\sigma\sqrt{T-t}} \tag{5.7}$$

Closely related to cash-or-nothing options are the asset-or-nothing call (AONC) and put (AONP). The holder of an asset-or-nothing call receives at expiry date T one unit of the underlying asset if its final price is larger than the strike price K, otherwise she gets nothing. The asset-or-nothing put follows the same rule, but it switches the payoffs. In other words,

$$AONC(T,x) = \begin{cases} x & \text{if} \quad x > K \\ 0 & \text{if} \quad x < K \end{cases} \tag{5.8}$$

$$AONP(T,x) = \begin{cases} 0 & \text{if} \quad x > K \\ x & \text{if} \quad x < K \end{cases} \tag{5.9}$$

We immediately observe that the sum of the final payoffs of a call and a put is, in any circumstances, the final price of the asset. Knowing that the sum of call and put valuations at any date has to coincide to the valuation of another derivative, it follows that this particular derivative is nothing else that the discounted underlying's price, thus we get a put-call parity relationship

$$AONC(t,x) + AONP(t,x) = x \exp[-q(T-t)] \tag{5.10}$$

In order to obtain a put-call symmetry relationship, let us define a symmetry

$$\Gamma := \Gamma_* \circ \Gamma_6(a'') \circ \Gamma_2(a') \circ \Gamma_4(a)$$

where

$$a = -2 \qquad a' = 2(\mathcal{D} + \sigma^2)T - \log K^2 = -a''$$

which can be described as follows

$$\Gamma \begin{cases} \hat{t} & = & t \\ \hat{x} & = & \frac{K^2}{x} \exp[-2(\mathcal{D}+\sigma^2)(T-t)] \\ \hat{u} & = & \frac{K^2}{x^2} u \exp[-2(\mathcal{D}+\sigma^2)(T-t)] \end{cases}$$

In other words, the newly constructed derivative instrument has a pricing formula

$$\Gamma[AONC(t,x)] = \frac{x^2}{K^2} \exp[4(\mathcal{D}+\sigma^2)(T-t)] AONC\left(t, \frac{K^2}{x} \exp[-2(\mathcal{D}+\sigma^2)(T-t)]\right)$$

We have to check the boundary condition

$$\Gamma[AONC(T,x)] = \begin{cases} \frac{x^2}{K^2} \cdot \frac{K^2}{x} & \text{if} \quad \frac{K^2}{x} > K \\ 0 & \text{if} \quad \frac{K^2}{x} < K \end{cases} = AONP(T,x)$$

We know that $\Gamma[AONC(t,x)]$ has to be the valuation of certain financial derivative that receives the same final payoff as $AONP(t,x)$. Therefore these two derivatives must coincide, hence we proved a put-call symmetry relationship

$$AONP(t,x) = \frac{x^2}{K^2}\exp[4(\mathcal{D}+\sigma^2)(T-t)]\cdot \tag{5.11}$$
$$\cdot AONC\left(t, \frac{K^2}{x}\exp[-2(\mathcal{D}+\sigma^2)(T-t)]\right)$$

Combining equations (5.10) and (5.11), we get the functional equation satisfied by AONC

$$AONC(t,x) + \frac{x^2}{K^2}\exp[4(\mathcal{D}+\sigma^2)(T-t)]\cdot \tag{5.12}$$
$$\cdot AONC\left(t, \frac{K^2}{x}\exp[-2(\mathcal{D}+\sigma^2)(T-t)]\right) = x\exp[-q(T-t)]$$

that leads to the solution

$$AONC(t,x) = x\exp[-q(T-t)]N(d^+) \tag{5.13}$$
$$AONP(t,x) = x\exp[-q(T-t)]N(-d^+)$$

where $N(\cdot)$ denotes the cumulative normal distribution function and

$$d^+ := d^- + \sigma\sqrt{T-t} = \frac{\log(x/K) + (\mathcal{D}+\sigma^2)(T-t)}{\sigma\sqrt{T-t}} \tag{5.14}$$

5.2 Gap and Vanilla Options

The binary options are used as building bricks to construct more complex financial derivatives. The best examples in this respect are the *gap options* GC and GP (Briys, Bellalah, Mai, de Varenne [8]), defined at expiry date as follows

$$GC(T,x) = \begin{cases} x-H & \text{if} \quad x > K \\ 0 & \text{if} \quad x < K \end{cases} \tag{5.15}$$

$$GP(T,x) = \begin{cases} 0 & \text{if} \quad x > K \\ H-x & \text{if} \quad x < K \end{cases} \tag{5.16}$$

One observes the occurrence of a secondary strike price H, the *gap* being defined as the difference $H-K$ between the two strike prices. In order to avoid a possible confusion, we have to specify the strike price(s) as parameter(s) when we talk about gap or binary options.

Adding together the final payoffs of a gap call with strikes (K, H) and H/K cash-or-nothing calls with strike K we get

$$GC(T, x; K, H) + \frac{H}{K} CONC(T, x; K) = \tag{5.17}$$

$$= \begin{cases} x - H & \text{if} \quad x > K \\ 0 & \text{if} \quad x < K \end{cases} + \frac{H}{K} \begin{cases} K & \text{if} \quad x > K \\ 0 & \text{if} \quad x < K \end{cases} =$$

$$= \begin{cases} x - H & \text{if} \quad x > K \\ 0 & \text{if} \quad x < K \end{cases} + \begin{cases} H & \text{if} \quad x > K \\ 0 & \text{if} \quad x < K \end{cases} = \begin{cases} x & \text{if} \quad x > K \\ 0 & \text{if} \quad x < K \end{cases}$$

which is exactly the final payoff for an asset-or-nothing call with strike price K. The argument that a positive multiple of a derivative valuation is still a derivative valuation (symmetry Γ_6) and that the set of derivative valuations is closed with respect to addition (Theorem 3.1) allows us to extend previous equation back in time

$$GC(t, x; K, H) + \frac{H}{K} CONC(t, x; K) = AONC(t, x; K) \tag{5.18}$$

and to obtain a formula for the valuation of a call gap

$$GC(t, x; K, H) = x \exp[-q(T - t)] N(d^+) - H \exp[-r(T - t)] N(d^-) \tag{5.19}$$

For puts, we first observe that

$$GP(T, x; K, H) + AONP(T, x; K) = \tag{5.20}$$

$$= \begin{cases} 0 & \text{if} \quad x > K \\ H - x & \text{if} \quad x < K \end{cases} + \begin{cases} 0 & \text{if} \quad x > K \\ x & \text{if} \quad x < K \end{cases} =$$

$$= \begin{cases} 0 & \text{if} \quad x > K \\ H & \text{if} \quad x < K \end{cases} = \frac{H}{K} \begin{cases} 0 & \text{if} \quad x > K \\ K & \text{if} \quad x < K \end{cases} = \frac{H}{K} CONP(T, x; K)$$

The same argument as before gives

$$GP(t, x; K, H) + AONP(t, x; K) = \frac{H}{K} CONP(t, x; K) \tag{5.21}$$

the formula for the valuation of a gap put being

$$GP(t, x; K, H) = H \exp[-r(T-t)] N(-d^-) - x \exp[-q(T-t)] N(-d^+) \tag{5.22}$$

A put-call parity relationship for gaps could be obtained from equations (5.4), (5.10), (5.18) and (5.21).

$$GC(t,x;K,H) - GP(t,x;K,H) = \tag{5.23}$$

$$= \left[AONC(t,x;K) - \frac{H}{K}CONC(t,x;K)\right] - \left[\frac{H}{K}CONP(t,x;K) - AONP(t,x;K)\right] =$$

$$= [AONC(t,x;K)+AONP(t,x;K)] - \frac{H}{K}[CONC(t,x;K)+CONP(t,x;K)] =$$

$$= x\exp[-q(T-t)] - H\exp[-r(T-t)]$$

When the gap vanishes (that is, when the strikes K and H coincide), the gap options GC and GP become the *European vanilla options* EC, respectively EP, with the same maturity date T and the same strike price K, as their payoffs

$$EC(T,x;K) = \begin{cases} x-K & \text{if} \quad x > K \\ 0 & \text{if} \quad x < K \end{cases} = \max(x-K,0) \tag{5.24}$$

$$EP(T,x;K) = \begin{cases} 0 & \text{if} \quad x > K \\ K-x & \text{if} \quad x < K \end{cases} = \max(K-x,0) \tag{5.25}$$

valuations formulae

$$EC(t,x;K) = x\exp[-q(T-t)]N(d^+) - K\exp[-r(T-t)]N(d^-) \tag{5.26}$$

$$EP(t,x;K) = K\exp[-r(T-t)]N(-d^-) - x\exp[-q(T-t)]N(-d^+) \tag{5.27}$$

and put-call parity relationship

$$EC(t,x;K) - EP(t,x;K) = x\exp[-q(T-t)] - K\exp[-r(T-t)] \tag{5.28}$$

show (Kwok [34], p. 70). Let us add that both vanilla European options can be constructed using only binaries.

$$EC(t,x;K) = AONC(t,x;K) - CONC(t,x;K) \tag{5.29}$$

$$EP(t,x;K) = CONP(t,x;K) - AONP(t,x;K) \tag{5.30}$$

5.3 Choosers

A *simple chooser option* (or called *as-you-like-it option*) entitles the holder to choose, at a predetermined date T_0 in the future, whether the option is a standard European call or put with a common maturity date T and a common strike price K. We usually denote the value of the simple chooser by

$$SC(t,x;T_0;T,K) \qquad T_0 < T$$

where t is the present date and x is the actual price of the underlying asset. Therefore the payoff from a simple chooser option at the choice date can be written as

$$SC(T_0,x;T_0;T,K) = \max[EC(T_0,x;T,K), EP(T_0,x;T,K)] \tag{5.31}$$

where the four arguments of each vanilla option respectively denote the present date, the actual underlying's price, the expiry date and the strike price. Using the put-call parity relationship (5.28), we may write

$$\begin{aligned} SC(T_0,x;T_0;T,K) &= \max\{EC(T_0,x;T,K), EC(T_0,x;T,K)+ \\ &\quad +K\exp[-r(T-T_0)] - x\exp[-q(T-T_0)]\} = \\ &= EC(T_0,x;T,K) + \max\{0, K\exp[-r(T-T_0)] - x\exp[-q(T-T_0)]\} = \\ &= EC(T_0,x;T,K) + \exp[-q(T-T_0)]\max\{0, K\exp[-(r-q)(T-T_0)] - x\} = \\ &= EC(T_0,x;T,K) + \Gamma_6(a_0)[EP(T_0,x;T_0,K_0)] \end{aligned} \tag{5.32}$$

where

$$a_0 := -q(T-T_0) \qquad K_0 := K\exp[-(r-q)(T-T_0)]$$

Extending this relationship back and forth in time, we get

$$\begin{aligned} SC(t,x;T_0;T,K) &= EC(t,x;T,K) + \Gamma_6(a_0)[EP(t,x;T_0,K_0)] = \\ &= x\exp[-q(T-t)]N(d^+) - K\exp[-r(T-t)]N(d^-)+ \\ &+\exp[-q(T-T_0)]\{K_0\exp[-r(T_0-t)]N(-d_*^-) - x\exp[-q(T_0-T)]N(-d_*^+)\} = \\ &= x\exp[-q(T-t)]N(d^+) - K\exp[-r(T-t)]N(d^-)+ \\ &+K\exp[-q(T-T_0)]\exp[-(r-q)(T-T_0)]\exp[-r(T_0-t)]N(-d_*^-)- \\ &-x\exp[-q(T-T_0)]\exp[-q(T_0-t)]N(-d_*^+) = \end{aligned} \tag{5.33}$$

$$= x\exp[-q(T-t)]N(d^+) - K\exp[-r(T-t)]N(d^-) +$$
$$+K\exp[-r(T-t)]N(-d_*^-) - x\exp[-q(T-t)]N(-d_*^+)$$

where

$$d_*^- := \frac{\log(x/K_0) + \mathcal{D}(T_0-t)}{\sigma\sqrt{T_0-t}} = \frac{\log(x/K) + (r-q)(T-T_0) + (r-q-\sigma^2/2)(T_0-t)}{\sigma\sqrt{T_0-t}} =$$

$$= \frac{\log(x/K) + (r-q)(T-t) - (\sigma^2/2)(T_0-t)}{\sigma\sqrt{T_0-t}}$$

$$d_*^+ := d_*^- + \sigma\sqrt{T_0-t} = \frac{\log(x/K) + (r-q)(T-t) + (\sigma^2/2)(T_0-t)}{\sigma\sqrt{T_0-t}}$$

as in Kwok [34], p. 82. Equation (5.33) shows that a simple chooser $SC(t,x;T_0;T,K)$ could be interpreted as a portfolio containing a vanilla call $EC(t,x;T,K)$ and some (more precisely $-q(T-T_0)$) vanilla puts $EP(t,x;T_0,K_0)$.

A *complex chooser option* generalizes the simple chooser by allowing the underlying vanilla call and put to have different strikes and maturity dates ($K_1,\ T_1$ for the call, $K_2,\ T_2$ for the put, $T_0 < T_1,\ T_2$). The valuation formula for a complex chooser was found by Rubinstein in 1991 (see Clewlow and Strickland [15], p. 56).

$$CC(t,x;T_0;T_1,K_1;T_2,K_2) = x\exp[-q(T_1-t)]N_2(d_0^+,d_1^+;\rho_1) - \qquad (5.34)$$
$$-K_1\exp[-r(T_1-t)]N_2(d_0^-,d_1^-;\rho_1) + K_2\exp[-r(T_2-t)]N_2(-d_0^-,-d_2^-;\rho_2) -$$
$$-x\exp[-q(T_2-t)]N_2(-d_0^+,-d_2^-;\rho_2)$$

where $N_2(\cdot,\cdot;\rho)$ denotes the standard bivariate normal cumulative distribution function with correlation coefficient ρ,

$$\rho_i := \sqrt{\frac{T_0-t}{T_i-t}} \qquad i = 1,2$$

$$d_i^- := \frac{\log(x/K_i) + \mathcal{D}(T_i-t)}{\sigma\sqrt{T_i-t}} \qquad d_i^+ := d_i^- + \sigma\sqrt{T_i-t} \qquad i = 0,1,2$$

and K_0 satisfies

$$EC(T_0,K_0;T_1,K_1) = EP(T_0,K_0;T_2,K_2)$$

Let us check that the complex chooser valuation reduces to the simple chooser valuation when $T_1 = T_2 = T$ and $K_1 = K_2 = K$. Observe first that the condition for K_0 becomes

$$EC(T_0, K_0; T, K) = EP(T_0, K_0; T, K) \Leftrightarrow$$

$$\Leftrightarrow K_0 \exp[-q(T-T_0)] = K \exp[-r(T-T_0)] \Leftrightarrow K_0 = K \exp[-(r-q)(T-T_0)]$$

Moreover

$$\rho_1 = \rho_2 = \rho = \sqrt{\frac{T_0 - t}{T - t}} \qquad d_0^- = d_*^- \quad d_0^+ = d_*^+$$

$$d_1^- = d_2^- = d^- \qquad d_1^+ = d_2^+ = d^+$$

Substitute everything into (5.34).

$$CC(t, x : T_0; T, K; T, K) = x \exp[-q(T-t)] N_2(d_*^+, d^+; \rho) -$$

$$-K \exp[-r(T-t)] N_2(d_*^-, d^-; \rho) + K \exp[-r(T-t)] N_2(-d_*^-, -d^-; \rho) -$$

$$-x \exp[-q(T-t)] N_2(-d_*^+, -d^+; \rho)$$

It is proved in Appendix E that

$$N_2(a, b; \rho) - N_2(-a, -b; \rho) = N(b) - N(-a) \qquad \forall a, b \in \mathbf{R} \quad \forall \rho \in (-1, 1)$$

hence

$$CC(t, x; T_0; T, K; T, K) = x \exp[-q(T-t)][N(d^+) - N(-d_*^+)] -$$

$$-K \exp[-r(T-t)][N(d^-) - N(-d_*^-)] = SC(t, x; T_0; T, K)$$

5.4 Compound Options

A compound option is an option on an option. The usual definition is a vanilla European put or call on a vanilla European put or call. This gives four basic compound options; a call on a call, a call on a put, a put on a call and a put on a put. Valuation formulae for all four variations were first published by Rubinstein (see Clewlow and Strickland, [15], p. 49). The expression for a call that has to be exercised at T_0 with a strike price of $\bar{K}$ on a call with parameters $T_1 > T_0$ and K_1 is

$$COC(t, x; T_0, \bar{K}; T_1, K_1) = x \exp[-q(T_1 - t)] N_2(d_*^+, d_1^+; \rho_1) - \qquad (5.35)$$

$$-K_1 \exp[-r(T_1 - t)] N_2(d_*^-, d_1^-; \rho_1) - \bar{K} \exp[-r(T_0 - t)] N(d_*^-)$$

where K_0 satisfies

$$EC(T_0, K_0; T_1, K_1) = \bar{K} \tag{5.36}$$

If the underlying is a put with parameters $T_2 > T_0$ and K_2, then the expression for a call on put is

$$COP(t, x; T_0, \bar{K}; T_2, K_2) = -x \exp[-q(T_2 - t)] N_2(-d_*^+, -d_2^+, \rho_2) + \tag{5.37}$$

$$+K_2 \exp[-r(T_2 - t)] N_2(-d_*^-, -d_2^-; \rho_2) - \bar{K} \exp[-r(T_0 - t)] N(-d_*^-)$$

where K_0 satisfies

$$EP(T_0, K_0; T_2, K_2) = \bar{K} \tag{5.38}$$

Suppose the common strike price $\bar{K}$ for a call on call and for a call on put has the property that there exists K_0 such that

$$EC(T_0, K_0; T_1, K_1) = EP(T_0, K_0; T_2, K_2) = \bar{K} \tag{5.39}$$

Using equations (5.34), (5.35) and (5.37) we get

$$COC(t, x; T_0, \bar{K}; T_1, K_1) + COP(t, x; T_0, \bar{K}; T_2, K_2) = \tag{5.40}$$

$$= CC(t, x; T_0; T_1, K_1; T_2, K_2) - \bar{K} \exp[-r(T_0 - t)]$$

This parity relationship deserves an explanation. Suppose that $T_0,\ T_1, T_2,\ \bar{K}$, $K_0,\ K_1,\ K_2$ satisfy all the conditions described above and consider a portfolio Π that contains a call on a call with parameters $T_0, \bar{K}; T_1, K_1$, a call on a put with parameters $T_0, \bar{K}; T_2, K_2$ and a riskless bond that pays an amount $\bar{K}$ of money at date T_0. Let, for any t and x, $\Pi(t, x)$ denote the value of portfolio Π at date t when the price of the underlying asset is x, and examine the value $\Pi(T_0, x)$. There are two cases.

- Suppose that the actual asset's price is lower than the cutoff level K_0. Since a vanilla call's value is increasing in underlying's price (Wilmott [48], p. 177), equation (5.39) shows that the call on call will not be exercised. A similar argument tells that the call on put will be exercised, getting a put with parameters T_2, K_2 in exchange on an amount $\bar{K}$ of money. The bond pays exactly the amount $\bar{K}$ that is required to purchase the put, so that the whole portfolio is reduced to the underlying put.

- Suppose that the actual asset price is larger than the cutoff level K_0. Then only the call on call is exercised, purchasing the underlying call with parameters T_1, K_1 in exchange of the bond's payoff.

Therefore the portfolio's value at T_0 coincides to the value of a complex chooser evaluated at the same date. We got

$$COC(T_0, x; T_0, \bar{K}; T_1, K_1) + COP(T_0, x; T_0, \bar{K}; T_2, K_2) + \\ + B(T_0; T_0, \bar{K}) = CC(T_0, x; T_0; T_1, K_1; T_2, K_2) \tag{5.41}$$

where $B(t; T_0, \bar{K})$ denotes the value of the bond at any date t. Extending in time last equation, it results

$$COC(t, x; T_0, \bar{K}; T_1, K_1) + COP(t, x; T_0, \bar{K}; T_2, K_2) + \\ + \bar{K} \exp[-r(T_0 - t)] = CC(t, x; T_0; T_1, K_1; T_2, K_2) \tag{5.42}$$

5.5 Barrier Options

Options with payoffs which depend on the complete path taken by the underlying's price to reach its expiration value are becoming increasingly popular. Some of the most widely used of these so-called *path dependent options* are the *barrier options*. These derivative instruments come in two types. A *knock-out option* cancels immediately when the underlying's price hits or crosses a predetermined level, typically referred to as the *barrier*. A *knock-in option* works the other way around. If the underlying's price does not hit or cross the barrier, the option does not come into existence and therefore expires worthless. Apart from distinguishing between knock-in and knock-out options, there is a second distinction to make. If the options knocks in or out when the underlying's price ends up above the barrier we speak of an *up-barrier*. Likewise, if the option knocks in or out when the underlying's price ends up below the barrier we speak of a *down-barrier*. Together this yields four basic types of barrier options: down-and-out (DO), up-and-out (UO), down-and-in (DI) and up-and-in (UI) options. In theory, barriers can be added to any existing type of ordinary or exotic option. In this section, however, we limit ourselves to vanilla call and put options.

All valuations formulae that follow are due to Reiner and Rubinstein [40] and are reproduced in Hull [28], p. 462. The notation follows that of Wilmott

[48], p. 247.

$$d_1 := \frac{\log(x/K) + (\mathcal{D} + \sigma^2)(T-t)}{\sigma\sqrt{T-t}}$$

$$d_2 := \frac{\log(x/K) + \mathcal{D}(T-t)}{\sigma\sqrt{T-t}}$$

$$d_3 := \frac{\log(x/H) + (\mathcal{D} + \sigma^2)(T-t)}{\sigma\sqrt{T-t}}$$

$$d_4 := \frac{\log(x/H) + \mathcal{D}(T-t)}{\sigma\sqrt{T-t}}$$

$$d_5 := \frac{\log(x/H) - \mathcal{D}(T-t)}{\sigma\sqrt{T-t}}$$

$$d_6 := \frac{\log(x/H) - (\mathcal{D} + \sigma^2)(T-t)}{\sigma\sqrt{T-t}}$$

$$d_7 := \frac{\log(xK/H^2) - \mathcal{D}(T-t)}{\sigma\sqrt{T-t}}$$

$$d_8 := \frac{\log(xK/H^2) - (\mathcal{D} + \sigma^2)(T-t)}{\sigma\sqrt{T-t}}$$

The parameters t, x, T, K have the same meaning as before, while H stands for the barrier level. Let us add that the eight barrier options can be divided into four pairs (down calls, up calls, down puts, up puts), each pair being endowed with a parity relationship

$$IN + OUT = VANILLA \tag{5.43}$$

Let us prove this formula. Consider a portfolio of one European in-option and one European out-option belonging to the same pair. Both have the same barrier, strike price and date of expiration, the sum of their values is simply the same of that of a corresponding vanilla option with the same strike price and expiration date. From financial arguments, this is obvious since only one of the two barrier options survives at expiry and either payoff is the same as that of the vanilla option.

5.5.1 Down Calls

Suppose that $K > H$. When the underlying's price x hits the barrier H from above, the down-and-in call (DIC) transforms itself into a vanilla call, that is,

$$DIC(t, H; T, K; H) = EC(t, H; T, K) \qquad \forall t \tag{5.44}$$

On the other hand, Appendix F shows that the pricing formula

$$\tilde{\Gamma}[EC(t, x; T, K)] := \left(\frac{H}{x}\right)^{\frac{2\mathcal{D}}{\sigma^2}} EC\left(t, \frac{H^2}{x}; T, K\right) \tag{5.45}$$

can be attached to a derivative instrument that satisfies

$$\tilde{\Gamma}[EC(t, H; T, K)] = EC(t, H; T, K) \tag{5.46}$$

Equations (5.44) and (5.46) show that DIC and $\tilde{\Gamma}(EC)$ have the same value along the line $x = H$, hence they coincide. Therefore we get

$$DIC(t, x; T, K; H) = \left(\frac{H}{x}\right)^{\frac{2\mathcal{D}}{\sigma^2}} EC\left(t, \frac{H^2}{x}; T, K\right) = \tag{5.47}$$

$$= \left(\frac{H}{x}\right)^{\frac{2\mathcal{D}}{\sigma^2}} \left[\frac{H^2}{x} \exp[-q(T-t)] N(-d_8) - K \exp[-r(T-t)] N(-d_7)\right]$$

The IN-OUT parity relationship allows us to deduce the valuation formula of a down-and-out call when $K > H$.

$$DOC(t, x; T, K; H) = EC(t, x; T, K) - DIC(t, x; T, K; H) = \tag{5.48}$$

$$= x \exp[-q(T-t)] N(d_1) - K \exp[-r(T-t)] N(d_2) -$$

$$- \left(\frac{H}{x}\right)^{\frac{2\mathcal{D}}{\sigma^2}} \left[\frac{H^2}{x} \exp[-q(T-t)] N(-d_8) - K \exp[-r(T-t)] N(-d_7)\right] =$$

$$= x \exp[-q(T-t)] \left[N(d_1) - \left(\frac{H}{x}\right)^{2+\frac{2\mathcal{D}}{\sigma^2}} N(-d_8)\right] -$$

$$- K \exp[-r(T-t)] \left[N(d_2) - \left(\frac{H}{x}\right)^{\frac{2\mathcal{D}}{\sigma^2}} N(-d_7)\right]$$

Suppose that $K < H$. When the underlying's price x hits the barrier H from above, the down-and-out call ceases to exist. Observe that the relationship

$$DOC(t,x;T,K;H) = GC(t,x;T,H,K) - \left(\frac{H}{x}\right)^{\frac{2\mathcal{D}}{\sigma^2}} GC\left(t,\frac{H^2}{x};T,H,K\right) \tag{5.49}$$

is satisfied along the line $x = H$, hence it remains valid for any x. Therefore the valuation of a down-and-out call becomes

$$\begin{aligned} DOC(t,x;T,K;H) &= x\exp[-q(T-t)]N(d_3) - K\exp[-r(T-t)]N(d_4) - \quad (5.50)\\ &- \left(\frac{H}{x}\right)^{\frac{2\mathcal{D}}{\sigma^2}} \left[\frac{H^2}{x}\exp[-q(T-t)]N(-d_6) - K\exp[-r(T-t)]N(-d_5)\right] \\ &= x\exp[-q(T-t)]\left[N(d_3) - \left(\frac{H}{x}\right)^{2+\frac{2\mathcal{D}}{\sigma^2}} N(-d_6)\right] - \\ &-K\exp[-r(T-t)]\left[N(d_4) - \left(\frac{H}{x}\right)^{\frac{2\mathcal{D}}{\sigma^2}} N(-d_5)\right] \end{aligned}$$

The corresponding formula for the down-and-in call when $K < H$ can be obtained using the IN-OUT parity relationship.

$$\begin{aligned} DIC(t,x;T,K;H) &= x\exp[-q(T-t)]\left[N(d_1) - N(d_3) + \left(\frac{H}{x}\right)^{2+\frac{2\mathcal{D}}{\sigma^2}} N(-d_6)\right] - \quad (5.51)\\ &-K\exp[-r(T-t)]\left[N(d_2) - N(d_4) + \left(\frac{H}{x}\right)^{\frac{2\mathcal{D}}{\sigma^2}} N(-d_5)\right] \end{aligned}$$

5.5.2 Up Calls

If $K > H$, the up-and-out call is valueless and the up-and-in call reduces itself to a vanilla call. Thus we can assume that $K < H$.

Let us check that the value of an up-and-in call coincide, along the line $x = H$, to the value of the following portfolio

$$GC(t,x;T,H,K) + \left(\frac{H}{x}\right)^{\frac{2\mathcal{D}}{\sigma^2}} \left[EP\left(t,\frac{H^2}{x};T,K\right) - GP\left(t,\frac{H^2}{x};T,H,K\right)\right] \tag{5.52}$$

Using the parity relationships for gap and vanilla options, we get the value of this portfolio for $x = H$

$$GC(t, H; T, H, K) + EP(t, H; T, K) - GP(t, H; T, H, K) =$$

$$= H \exp[-q(T-t)] - K \exp[-r(T-t)] + EP(t, H; T, K) = EC(t, H; T, K)$$

which is exactly the value that an up-and-in call gets when it crosses the barrier from below. Therefore we showed that for any x and t the following equality holds.

$$UIC(t, x; T, K; H) = \left(\frac{H}{x}\right)^{\frac{2\mathcal{D}}{\sigma^2}} \left[EP\left(t, \frac{H^2}{x}; T, K\right) - GP\left(t, \frac{H^2}{x}; T, H, K\right)\right] + \tag{5.53}$$
$$+GC(t, x; T, H, K) = x \exp[-q(T-t)] \left\{N(d_3) + \left(\frac{H}{x}\right)^{2+\frac{2\mathcal{D}}{\sigma^2}} [N(d_6) - N(d_8)]\right\} -$$
$$-K \exp[-r(T-t)] \left\{N(d_4) + \left(\frac{H}{x}\right)^{\frac{2\mathcal{D}}{\sigma^2}} [N(d_5) - N(d_7)]\right\}$$

IN-OUT parity gives

$$UOC(t, x; T, K; H) = x \exp[-q(T-t)] \left\{N(d_1) - N(d_3) - \left(\frac{H}{x}\right)^{2+\frac{2\mathcal{D}}{\sigma^2}} [N(d_6) - N(d_8)]\right\} - \tag{5.54}$$
$$-K \exp[-r(T-t)] \left\{N(d_2) - N(d_4) - \left(\frac{H}{x}\right)^{\frac{2\mathcal{D}}{\sigma^2}} [N(d_5) - N(d_7)]\right\}$$

5.5.3 Down Puts

If $K < H$, the down-and-out put is valueless and the down-and-in put reduces itself to a vanilla put. Thus we can assume that $K > H$.

Let us check that the value of an down-and-in put coincide, along the line $x = H$, to the value of the following portfolio

$$GP(t, x; T, H, K) + \left(\frac{H}{x}\right)^{\frac{2\mathcal{D}}{\sigma^2}} \left[EC\left(t, \frac{H^2}{x}; T, K\right) - GC\left(t, \frac{H^2}{x}; T, H, K\right)\right] \tag{5.55}$$

Using the parity relationships for gap and vanilla options, we get the value of this portfolio for $x = H$

$$GP(t,H;T,H,K) + EC(t,H;T,K) - GC(t,H;T,H,K) =$$
$$= -H\exp[-q(T-t)] + K\exp[-r(T-t)] + EC(t,H;T,K) = EP(t,H;T,K)$$

which is exactly the value that an down-and-in put gets when it crosses the barrier from above. Therefore we showed that for any x and t the following equality holds.

$$DIP(t,x;T,K;H) = \left(\frac{H}{x}\right)^{\frac{2\mathcal{D}}{\sigma^2}} \left[EC\left(t,\frac{H^2}{x};T,K\right) - GC\left(t,\frac{H^2}{x};T,H,K\right)\right] + \tag{5.56}$$
$$+GP(t,x;T,H,K) = -x\exp[-q(T-t)]\left\{N(-d_3) + \left(\frac{H}{x}\right)^{2+\frac{2\mathcal{D}}{\sigma^2}}[N(d_8) - N(d_6)]\right\} +$$
$$+K\exp[-r(T-t)]\left\{N(-d_4) + \left(\frac{H}{x}\right)^{\frac{2\mathcal{D}}{\sigma^2}}[N(d_7) - N(d_5)]\right\}$$

IN-OUT parity gives

$$DOP(t,x;T,K;H) = x\exp[-q(T-t)]\left\{N(d_1) - N(d_3) - \left(\frac{H}{x}\right)^{2+\frac{2\mathcal{D}}{\sigma^2}}[N(d_6) - N(d_8)]\right\} - \tag{5.57}$$
$$-K\exp[-r(T-t)]\left\{N(d_2) - N(d_4) - \left(\frac{H}{x}\right)^{\frac{2\mathcal{D}}{\sigma^2}}[N(d_5) - N(d_7)]\right\}$$

5.5.4 Up Puts

Suppose that $K > H$. When the underlying's price x hits the barrier H from below, the up-and-in put transforms itself into a vanilla put, hence the relationship

$$UIP(t,x;T,K;H) = \left(\frac{H}{x}\right)^{\frac{2\mathcal{D}}{\sigma^2}} EP\left(t,\frac{H^2}{x};T,K\right) \tag{5.58}$$

holds along the line $x = H$. Extending it to any x we get

$$UIP(t,x;T,K;H) = \left(\frac{H}{x}\right)^{\frac{2\mathcal{D}}{\sigma^2}}\left[K\exp[-r(T-t)]N(d_7) - \frac{H^2}{x}\exp[-q(T-t)]N(d_8)\right] \tag{5.59}$$

The IN-OUT parity relationship allows us to deduce the valuation formula of a up-and-out put when $K > H$.

$$UOP(t,x;T,K;H) = EP(t,x;T,K) - UIP(t,x;T,K;H) = \tag{5.60}$$
$$= K\exp[-r(T-t)][N(-d_2) - \left(\frac{H}{x}\right)^{\frac{2\mathcal{D}}{\sigma^2}} N(d_7)] -$$
$$-x\exp[-q(T-t)]\left[N(-d_1) - \left(\frac{H}{x}\right)^{2+\frac{2\mathcal{D}}{\sigma^2}} N(d_8)\right]$$

Suppose that $K < H$. When the underlying's price x hits the barrier H from below, the up-and-out put ceases to exist. Observe that the relationship

$$UOP(t,x;T,K;H) = GP(t,x;T,H,K) - \left(\frac{H}{x}\right)^{\frac{2\mathcal{D}}{\sigma^2}} GP\left(t,\frac{H^2}{x};T,H,K\right) \tag{5.61}$$

is satisfied along the line $x = H$, hence it remains valid for any x. Therefore the valuation of a up-and-out put becomes

$$UOP(t,x;T,K;H) = K\exp[-r(T-t)]N(-d_4) - x\exp[-q(T-t)]N(-d_3) + \tag{5.62}$$
$$+\left(\frac{H}{x}\right)^{\frac{2\mathcal{D}}{\sigma^2}}\left[\frac{H^2}{x}\exp[-q(T-t)]N(d_6) - K\exp[-r(T-t)]N(d_5)\right] =$$
$$= K\exp[-r(T-t)]\left[N(-d_4) - \left(\frac{H}{x}\right)^{\frac{2\mathcal{D}}{\sigma^2}} N(d_5)\right] -$$
$$-x\exp[-q(T-t)]\left[N(-d_3) - \left(\frac{H}{x}\right)^{2+\frac{2\mathcal{D}}{\sigma^2}} N(d_6)\right]$$

The corresponding formula for the up-and-in put when $K < H$ can be obtained using the IN-OUT parity relationship.

$$UIP(t,x;T,K;H) = K\exp[-r(T-t)]\left[N(d_4) - N(d_2) + \left(\frac{H}{x}\right)^{\frac{2\mathcal{D}}{\sigma^2}} N(d_5)\right] - \tag{5.63}$$
$$-x\exp[-q(T-t)]\left[N(d_3) - N(d_1) + \left(\frac{H}{x}\right)^{2+\frac{2\mathcal{D}}{\sigma^2}} N(d_6)\right]$$

Chapter 6

Conclusions

The thesis contains two types of results. As far as Mathematics is concerned, it is the first time when all the discrete symmetries of the Black-Scholes equation are explicitly computed and the corresponding full symmetry group is determined (equations (3.11)-(3.14)). The method used in these computations, namely the construction of the automorphism group associated to equation's Lie algebra (section 2.5), is original too.

From the point of view of Finance, there are many derivative pricing formulae proved using symmetry analysis. First we discovered put-call symmetry relationships for both types of binary options (equations (5.3) and (5.11)), then we derived for them certain functional equations (5.5) and (5.12) that lead to the well-known formulae (5.6), respectively (5.13). Using the binary options as building bricks, we constructed pricing formulae for gap and vanilla options (equations (5.19), (5.22), (5.26), (5.27)), for simple and complex choosers (equations (5.33) and (5.34)). Finally, we hedged the eight types of barrier options with European gaps and vanillas getting equations (5.47)-(5.63).

We should stress that all valuation formulae we derived are not new, they were all discovered before. However, only the symmetry analysis framework allows to construct such a large number of pricing formulae and parity relationships in a unitary manner. Furthermore, the same tools helped us to derive a couple of original symmetry relationships involving derivative securities' pricing formulae.

Chapter 7

Ideas for Future Research

The method described in the previous chapters could be used to compute symmetries of more general partial differential equations corresponding to various complex financial derivatives. Let us consider the example of the *Asian options*. Asian options are averaging instruments where the terminal payoffs depend on some form of averaging of the price of the underlying asset over a part or the whole life of the option. Most Asian options are of European-type since an Asian option of American-type may be redeemed as early as the start of the averaging period and lose the intent of protection from averaging.

There are two main classes of Asian options, namely the *fixed strike options* and the *floating strike options*. The terminal payoff functions of a fixed strike call option and a floating strike call option are $\max(y_T - K, 0)$ and $\max(x_T - y_T, 0)$, respectively, where x_T is the asset price at expiry, K is the strike price, and y denotes some form of average of the price of the underlying asset. The value of y depends on the path followed by the asset price.

If we take y to represent the *continuous sum*

$$y = \int_{t_0}^{T} x(\tau) d\tau$$

where $[t_0, T]$ is the averaging interval, then the PDE satisfied by the Asian options is (see Kwok [34], page 285)

$$\frac{\partial u}{\partial t} + \frac{\sigma^2 x^2}{2} \frac{\partial^2 u}{\partial x^2} + (r - q)x \frac{\partial u}{\partial x} + x \frac{\partial u}{\partial y} - ru = 0 \tag{7.1}$$

If y denotes the *continuous arithmetic average*

$$y = \frac{1}{T - t_0} \int_{t_0}^{T} x(\tau) d\tau$$

then the corresponding PDE is (Kwok [34], page 286)

$$\frac{\partial u}{\partial t} + \frac{\sigma^2 x^2}{2} \frac{\partial^2 u}{\partial x^2} + (r - q) x \frac{\partial u}{\partial x} + \frac{x}{T - t_0} \frac{\partial u}{\partial y} - ru = 0 \tag{7.2}$$

We could define the *continuous arithmetic running average*

$$y(t) = \frac{1}{t - t_0} \int_{t_0}^{t} x(\tau) d\tau$$

for any $t \in [t_0, T]$, where, by continuity, $y(t_0) = x(t_0)$. The PDE satisfied by the Asian options becomes (see Barraquand and Pudet [4])

$$\frac{\partial u}{\partial t} + \frac{\sigma^2 x^2}{2} \frac{\partial^2 u}{\partial x^2} + (r - q) x \frac{\partial u}{\partial x} + \frac{x - y}{T - t_0} \frac{\partial u}{\partial y} - ru = 0 \tag{7.3}$$

Any of the previous three equations is satisfied not only by the Asian options, but also by the European vanilla options. Henderson, Hobson, Shaw and Wojakowski defined in their paper [26] a *generalized Asian option* as a security that satisfies equation (7.3) and pays

$$\max(a x_T + b y_T + c, 0)$$

at expiry. Some remarkable particular cases:

- $(a, b, c) = (0, 1, -K)$ fixed-strike Asian call;
- $(a, b, c) = (0, -1, K)$ fixed-strike Asian put;
- $(a, b, c) = (1, -1, 0)$ floating-strike Asian call;
- $(a, b, c) = (-1, 1, 0)$ floating-strike Asian put;
- $(a, b, c) = (1, 0, -K)$ European call;
- $(a, b, c) = (-1, 0, K)$ European put;

Therefore one could compute the symmetries (both continuous and discrete) of equation (7.3) and translate them into relationships linking together two or more equation's solutions. Four types of significant results could be obtained:

1. Relationships involving only European options.
2. Relationships between Asian options of the same type (fixed-strike or floating-strike).
3. Relationships between different types of Asians.
4. Relationships between Europeans and Asians.

In the first two cases we would probably only get a couple of rather trivial put-call relationships, all of them well known.

Only one relationship involving Europeans and Asians was discovered by now, a parity shown by Alziary, Decamps and Koehl [1] that involves all the four Asians and both Europeans. It is a direct consequence of three put-call parity relationships written respectively for Europeans, for fixed-strike Asians and for floating-strike Asians, plus a symmetry reflecting the linearity of equation (7.3). There is some hope in employing symmetry analysis to discover new and less trivial relationships between Europeans and Asians.

One encounters a different situation when one looks at the "state-of-the-art" of the research concerning third type relationships. Henderson and Wojakowski proved in [25], using martingale theory, a symmetry (in fact, a pair of symmetries) between the price of a floating-strike Asian and the price of a related fixed-strike Asian, shown to be valid at the start of the averaging period. The existence of the symmetry was signalled before by Hoogland and Neumann in [27] and it was extended by Eberlein and Papapantoleon in [19] to cover exponential Lévy models of the asset price. Henderson and Wojakowski considered a fixed-strike call $c_x(K, x_0, r, q, 0, T)$ and a floating-strike call $c_f(x_0, \lambda, r, q, 0, T)$ evaluated at the beginning of the averaging period $[0, T]$ together with the corresponding puts $p_x(K, x_0, r, q, 0, T)$ and $p_f(x_0, \lambda, r, q, 0, T)$. Their maturity payoffs are respectively

$$c_x(K, x_T, r, q, 0, T) = \max(y_T - K, 0), \quad p_x(K, x_T, r, q, 0, T) = \max(K - y_T, 0)$$

$$c_f(x_T, \lambda, r, q, 0, T) = \max(\lambda x_T - y_T, 0), \quad p_f(x_T, \lambda, r, q, 0, T) = \max(y_T - \lambda x_T, 0)$$

where λ is a positive weighting factor. Under these assumptions, the following symmetry relationships hold:

$$c_f(x_0, \lambda, p, q, 0, T) = p_x(\lambda x_0, x_0, q, r, 0, T)$$

$$c_x(K, x_0, r, q, 0, T) = p_f(x_0, \frac{K}{x_0}, q, r, 0, T)$$

The last equation looks very much like the $\tilde{\Gamma}$ symmetry described in Appendix F and this remark gives us hopes in proving/extending Henderson and Wojakowski's results via symmetry analysis.

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Appendix A

Solving the Determining System

A.1 First Determining Equation

We observe that one must have $\xi^1(0) = 0$. Assuming $x \neq 0$, equation

$$x\left(\xi^1_x - \frac{\xi^0_t}{2}\right) = \xi^1 \tag{A.1}$$

can be written as follows:

$$\frac{x\xi^1_x - \xi^1}{x^2} = \frac{\xi^0_t}{2x} \tag{A.2}$$

By integration with respect to x we get

$$\frac{\xi^1}{x} = \frac{\xi^0_t}{2}\log|x| + \tau(t) \tag{A.3}$$

where $\tau(\cdot)$ is any smooth function of time. Finally we obtain the general form of function $\xi^1(\cdot,\cdot)$

$$\xi^1 = \frac{\xi^0_t}{2}x\log|x| + \tau(t)x \tag{A.4}$$

which still holds when $x = 0$. Its relevant partial derivatives are

$$\xi^1_x = \frac{\xi^0_t}{2}(1 + \log|x|) + \tau$$

$$\xi^1_{xx} = \frac{\xi^0_t}{2x}$$

$$\xi^1_t = \frac{\xi^0_{tt}}{2} x \log|x| + \tau_t x$$

A.2 Second Determining Equation

After substituting all the partial derivatives of ξ^1, the second determining equation becomes

$$\frac{\mathcal{D}}{2}\xi^0_t x = -\sigma^2 x^2 \beta_x + \frac{\xi^0_{tt} x \log|x|}{2} + \tau_t x \tag{A.5}$$

or, equivalently,

$$\beta_x = \frac{1}{\sigma^2}\left(\frac{\xi^0_{tt}\log|x|}{2x} + \frac{2\tau_t - \mathcal{D}\xi^0_t}{2x}\right) \tag{A.6}$$

Integration with respect to x gives

$$\beta = \frac{1}{\sigma^2}\left[\frac{\xi^0_{tt}\log^2|x|}{4} + \frac{(2\tau_t - \mathcal{D}\xi^0_t)\log|x|}{2}\right] + \omega(t) \tag{A.7}$$

where ω is any smooth function of time. We have to compute the relevant partial derivatives of function $\beta(\cdot,\cdot)$

$$\beta_{xx} = \frac{1}{\sigma^2}\left(\frac{\xi^0_{tt}(1-\log|x|)}{2x^2} - \frac{2\tau_t - \mathcal{D}\xi^0_t}{2x^2}\right) = \frac{1}{\sigma^2}\left(\frac{\xi^0_{tt} + \mathcal{D}\xi^0_t - 2\tau_t}{2x^2} - \frac{\xi^0_{tt}\log|x|}{2x^2}\right)$$

$$\beta_t = \frac{1}{\sigma^2}\left[\frac{\xi^0_{ttt}\log^2|x|}{4} + \frac{(2\tau_{tt} - \mathcal{D}\xi^0_{tt})\log|x|}{2}\right] + \omega_t$$

A.3 Third Determining Equation

$$\frac{1}{\sigma^2}\left[\frac{\xi^0_{ttt}\log^2|x|}{4} + \frac{(2\tau_{tt} - \mathcal{D}\xi^0_{tt})\log|x|}{2}\right] + \omega_t = \tag{A.8}$$

$$= \frac{\xi^0_{tt}\log|x|}{4} - \frac{\xi^0_{tt} + \mathcal{D}\xi^0_t - 2\tau_t}{4} - \frac{r-q}{\sigma^2}\left(\frac{\xi^0_{tt}\log|x|}{2} + \frac{2\tau_t - \mathcal{D}\xi^0_t}{2}\right) + r\xi^0_t$$

After some re-arrangements we get

$$\frac{\xi^0_{ttt}}{4\sigma^2}\log^2|x| + \frac{\tau_{tt}}{\sigma^2}\log|x| + \omega_t = -\frac{\xi^0_{tt}}{4} + \left(\frac{\mathcal{D}^2}{2\sigma^2} + r\right)\xi^0_t - \frac{\mathcal{D}}{\sigma^2}\tau_t \tag{A.9}$$

Identifying the corresponding coefficients, we arrive to the following system

$$\begin{cases} \xi^0_{ttt} &= 0 \\ \tau_{tt} &= 0 \\ \omega_t &= -\frac{\xi^0_{tt}}{4} + \left(\frac{\mathcal{D}^2}{2\sigma^2} + r\right)\xi^0_t - \frac{\mathcal{D}}{\sigma^2}\tau_t \end{cases} \tag{A.10}$$

A.4 Fourth Determining Equation

It only shows that $\alpha(\cdot,\cdot)$ is a solution to the Black-Scholes equation.

A.5 Determining System's General Solution

Equation (A.10) gives

$$\begin{cases} \xi^0 &= C_1 + C_2 t + C_3 t^2 \\ \tau &= C_4 + C_5 t \\ \omega &= \left(\frac{\mathcal{D}^2}{2\sigma^2} + r\right)C_3 t^2 + \left[\left(\frac{\mathcal{D}^2}{2\sigma^2} + r\right)C_2 - \frac{C_3}{2} - \frac{\mathcal{D}}{\sigma^2}C_5\right]t + C_6 \end{cases} \tag{A.11}$$

where $C_1, C_2, \dots, C_6$ are arbitrary constants. Equation (A.7) implies that

$$\beta = \frac{1}{\sigma^2}\left\{\frac{C_3\log^2|x|}{2} + \frac{[2C_5 - \mathcal{D}(C_2 + 2C_3 t)]\log|x|}{2}\right\} + \omega \tag{A.12}$$

where $\omega(\cdot)$ is given by equation (A.11). Function ξ^1 can be obtained from equation (A.4).

$$\xi^1 = \frac{(C_2 + 2C_3 t)x\log|x|}{2} + (C_4 + C_5 t)x \tag{A.13}$$

The general solution to the determining system can be described as follows

$$\left\{\begin{array}{lcl}\xi^0 & = & C_1 + C_2 t + C_3 t^2 \\ \xi^1 & = & \frac{(C_2+2C_3 t)x\log|x|}{2} + (C_4 + C_5 t)x \\ \eta & = & \alpha + \beta u \\ \beta & = & \frac{1}{\sigma^2}\left\{\frac{C_3\log^2|x|}{2} + \frac{[2C_5 - \mathcal{D}(C_2+2C_3 t)]\log|x|}{2}\right\} + \omega \\ \omega & = & \left(\frac{\mathcal{D}^2}{2\sigma^2} + r\right) C_3 t^2 + \left[\left(\frac{\mathcal{D}^2}{2\sigma^2} + r\right) C_2 - \frac{C_3}{2} - \frac{\mathcal{D}}{\sigma^2} C_5\right] t + C_6\end{array}\right. \tag{A.14}$$

where $C_1, C_2, \ldots, C_6$ are arbitrary constants and α is any solution to the standard Black-Scholes equation.

Appendix B

The Automorphism Group of $\mathcal{L}$

Let $\theta : \mathcal{L} \to \mathcal{L}$ be an automorphism. Then it has to preserve the center $Z(\mathcal{L}) = \langle \mathbf{v}_6 \rangle$, hence there exists a nonzero number Δ such that $\theta(\mathbf{v}_6) = \Delta \mathbf{v}_6$. Denote $\delta := \sqrt{|\Delta|}$ and $\varepsilon := \mathrm{sgn}(\Delta)$, so $\theta(\mathbf{v}_6) = \varepsilon\delta^2$. Any automorphism must preserve the radical $\mathcal{R} = \langle \mathbf{v}_2,\ \mathbf{v}_4,\ \mathbf{v}_6 \rangle$, hence there exist certain constants $b_{22}, b_{24}, b_{26}, b_{42}, b_{44}, b_{46}$ such that

$$\theta(\mathbf{v}_2) = b_{22}\mathbf{v}_2 + b_{24}\mathbf{v}_4 + b_{26}\mathbf{v}_6 \tag{B.1}$$

$$\theta(\mathbf{v}_4) = b_{42}\mathbf{v}_2 + b_{44}\mathbf{v}_4 + b_{46}\mathbf{v}_6$$

where the coefficients satisfy

$$b_{22}b_{44} - b_{24}b_{42} = \varepsilon\delta^2. \tag{B.2}$$

If we pre-multiply θ by $\exp(\lambda \mathrm{ad}(\mathbf{v}_1))$, the resulting automorphism $\bar{\theta}$ satisfies

$$\bar{\theta}(\mathbf{v}_2) = (b_{22} - \lambda b_{24})\mathbf{v}_2 + b_{24}\mathbf{v}_4 + b_{26}\mathbf{v}_6 \tag{B.3}$$

$$\bar{\theta}(\mathbf{v}_4) = (b_{42} - \lambda b_{44})\mathbf{v}_2 + b_{44}\mathbf{v}_4 + b_{46}\mathbf{v}_6$$

Case 1 $b_{44} \neq 0$. We choose $\lambda := b_{42}/b_{44}$, then we pre-multiply $\bar{\theta}$ by $\exp(\lambda \mathrm{ad}(\mathbf{v}_3))$, where $\lambda := \log(|b_{44}|/\delta)$, the resulting automorphism $\tilde{\theta}$ satisfies

$$\tilde{\theta}(\mathbf{v}_2) = \varepsilon\varepsilon'\delta\mathbf{v}_2 + \frac{b_{24}\delta}{|b_{44}|}\mathbf{v}_4 + b_{26}\mathbf{v}_6 \tag{B.4}$$

$$\tilde{\theta}(\mathbf{v}_4) = \varepsilon'\delta\mathbf{v}_4 + b_{46}\mathbf{v}_6$$

where $\varepsilon' := \mathrm{sgn}(b_{44})$. Next we can get rid of the coefficients b_{24}, b_{26}, b_{46} pre-multiplying $\widetilde{\theta}$ by appropriate inner automorphisms induced by $\mathbf{v}_5, \mathbf{v}_4, \mathbf{v}_2$. Finally we get that when $b_{44} \neq 0$, the automorphism θ is equivalent to an automorphism φ that satisfies:

$$\varphi(\mathbf{v}_2) = \varepsilon\varepsilon'\delta\mathbf{v}_2 \tag{B.5}$$

$$\varphi(\mathbf{v}_4) = \varepsilon'\delta\mathbf{v}_4$$

Automorphism φ has to preserve the centralizer structure, that is,

$$\langle\varphi(\mathbf{v}_1), \varphi(\mathbf{v}_2), \varphi(\mathbf{v}_6)\rangle = \varphi(C_{\mathcal{L}}(\mathbf{v}_2)) = C_{\mathcal{L}}(\varphi(\mathbf{v}_2)) = C_{\mathcal{L}}(\mathbf{v}_2) = \langle\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_6\rangle$$

$$\langle\varphi(\mathbf{v}_4), \varphi(\mathbf{v}_5), \varphi(\mathbf{v}_6)\rangle = \varphi(C_{\mathcal{L}}(\mathbf{v}_4)) = C_{\mathcal{L}}(\varphi(\mathbf{v}_4)) = C_{\mathcal{L}}(\mathbf{v}_4) = \langle\mathbf{v}_4, \mathbf{v}_5, \mathbf{v}_6\rangle$$

hence it satisfies

$$\varphi(\mathbf{v}_1) = b_{11}\mathbf{v}_1 + b_{12}\mathbf{v}_2 + b_{16}\mathbf{v}_6$$

$$\varphi(\mathbf{v}_5) = b_{54}\mathbf{v}_4 + b_{55}\mathbf{v}_5 + b_{56}\mathbf{v}_6$$

$$\varphi(\mathbf{v}_3) = \varphi([\mathbf{v}_1, \mathbf{v}_5]) = b_{11}b_{54}\mathbf{v}_2 + b_{11}b_{55}\mathbf{v}_3 + b_{12}b_{55}\mathbf{v}_4 + b_{12}b_{54}\mathbf{v}_6$$

Equations $[\mathbf{v}_1, \mathbf{v}_4] = \mathbf{v}_2,\ [\mathbf{v}_2, \mathbf{v}_5] = \mathbf{v}_4,\ [\mathbf{v}_1, \mathbf{v}_3] = 2\mathbf{v}_1,\ [\mathbf{v}_3, \mathbf{v}_5] = 2\mathbf{v}_5$ give $b_{12} = b_{54} = b_{16} = b_{56} = 0$ and $b_{11} = b_{55} = \varepsilon$. Automorphism φ gets a simple form:

$\mathbf{v}$	$\mathbf{v}_1$	$\mathbf{v}_2$	$\mathbf{v}_3$	$\mathbf{v}_4$	$\mathbf{v}_5$	$\mathbf{v}_6$
$\varphi(\mathbf{v})$	$\varepsilon\mathbf{v}_1$	$\varepsilon\varepsilon'\delta\mathbf{v}_2$	$\mathbf{v}_3$	$\varepsilon'\delta\mathbf{v}_4$	$\varepsilon\mathbf{v}_5$	$\varepsilon\delta^2\mathbf{v}_6$

Case 2 $b_{44} = 0$. As in the previous case, we can prove that automorphism θ is equivalent to an automorphism ψ, that satisfies:

$\mathbf{v}$	$\mathbf{v}_1$	$\mathbf{v}_2$	$\mathbf{v}_3$	$\mathbf{v}_4$	$\mathbf{v}_5$	$\mathbf{v}_6$
$\psi(\mathbf{v})$	$\varepsilon\mathbf{v}_5$	$-\varepsilon\varepsilon'\delta\mathbf{v}_4$	$-\mathbf{v}_3$	$\varepsilon'\delta\mathbf{v}_2$	$\varepsilon\mathbf{v}_1$	$\varepsilon\delta^2\mathbf{v}_6$

where $\varepsilon' := \mathrm{sgn}(b_{42})$. The set $\mathcal{G} = \{\varphi_{\varepsilon\varepsilon'}(\delta), \psi_{\varepsilon\varepsilon'}(\delta) |\ \varepsilon, \varepsilon' \in \{-1, 1\}, \delta > 0\}$ of the automorphisms constructed above possesses the property that any automorphism of the Lie algebra $\mathcal{L}$ is equivalent, modulo $\mathrm{Inn}(\mathcal{L})$, to one element of $\mathcal{G}$. Moreover, Remark 2.1 shows that the automorphisms in $\mathcal{G}$ are pairwise non-equivalent. Therefore $\mathcal{G}$ represents a complete set of representatives of Aut($\mathcal{L}$) modulo Inn($\mathcal{L}$).

One observes that $(\mathcal{G}, \circ)$ is a group itself, its multiplication table being:

$\circ$	$\varphi_{++}(\rho)$	$\varphi_{+-}(\rho)$	$\varphi_{-+}(\rho)$	$\varphi_{--}(\rho)$	$\psi_{++}(\rho)$	$\psi_{+-}(\rho)$	$\psi_{-+}(\rho)$	$\psi_{--}(\rho)$
$\varphi_{++}(\pi)$	$\varphi_{++}(\delta)$	$\varphi_{+-}(\delta)$	$\varphi_{-+}(\delta)$	$\varphi_{--}(\delta)$	$\psi_{++}(\delta)$	$\psi_{+-}(\delta)$	$\psi_{-+}(\delta)$	$\psi_{--}(\delta)$
$\varphi_{+-}(\pi)$	$\varphi_{+-}(\delta)$	$\varphi_{++}(\delta)$	$\varphi_{--}(\delta)$	$\varphi_{-+}(\delta)$	$\psi_{+-}(\delta)$	$\psi_{++}(\delta)$	$\psi_{--}(\delta)$	$\psi_{-+}(\delta)$
$\varphi_{-+}(\pi)$	$\varphi_{-+}(\delta)$	$\varphi_{--}(\delta)$	$\varphi_{++}(\delta)$	$\varphi_{+-}(\delta)$	$\psi_{--}(\delta)$	$\psi_{-+}(\delta)$	$\psi_{+-}(\delta)$	$\psi_{++}(\delta)$
$\varphi_{--}(\pi)$	$\varphi_{--}(\delta)$	$\varphi_{-+}(\delta)$	$\varphi_{+-}(\delta)$	$\varphi_{++}(\delta)$	$\psi_{-+}(\delta)$	$\psi_{--}(\delta)$	$\psi_{++}(\delta)$	$\psi_{+-}(\delta)$
$\psi_{++}(\pi)$	$\psi_{++}(\delta)$	$\psi_{+-}(\delta)$	$\psi_{-+}(\delta)$	$\psi_{--}(\delta)$	$\varphi_{+-}(\delta)$	$\varphi_{++}(\delta)$	$\varphi_{--}(\delta)$	$\varphi_{-+}(\delta)$
$\psi_{+-}(\pi)$	$\psi_{+-}(\delta)$	$\psi_{++}(\delta)$	$\psi_{--}(\delta)$	$\psi_{-+}(\delta)$	$\varphi_{++}(\delta)$	$\varphi_{+-}(\delta)$	$\varphi_{-+}(\delta)$	$\varphi_{--}(\delta)$
$\psi_{-+}(\pi)$	$\psi_{-+}(\delta)$	$\psi_{--}(\delta)$	$\psi_{++}(\delta)$	$\psi_{+-}(\delta)$	$\varphi_{-+}(\delta)$	$\varphi_{--}(\delta)$	$\varphi_{++}(\delta)$	$\varphi_{+-}(\delta)$
$\psi_{--}(\pi)$	$\psi_{--}(\delta)$	$\psi_{-+}(\delta)$	$\psi_{+-}(\delta)$	$\psi_{++}(\delta)$	$\varphi_{--}(\delta)$	$\varphi_{-+}(\delta)$	$\varphi_{+-}(\delta)$	$\varphi_{++}(\delta)$

where $\delta := \pi\rho$. Hence, $\mathcal{G}$ is isomorphic to the *outer automorphism group* of the Lie algebra $\mathcal{L}$. Its subgroups $\mathcal{H} := \{\varphi_{\varepsilon\varepsilon'}(1), \psi_{\varepsilon\varepsilon'}(1) |\ \varepsilon, \varepsilon' \in \{-1, 1\}\}$ and $\mathcal{K} := \{\varphi_{++}(\delta) |\ \delta > 0\}$ are respectively isomorphic to the dihedral group D_8 and to the multiplicative group of the positive numbers. Moreover, $\mathcal{G}$ is the direct product of its subgroups $\mathcal{H}$ and $\mathcal{K}$. All this remarks are helping us to establish the structure of the full automorphism group Aut($\mathcal{L}$).

Proposition B.1 *The outer automorphism group Out($\mathcal{L}$) is isomorphic to the direct product $D_8 \times (0, \infty)$.*

Corollary B.1 *The full automorphism group Aut($\mathcal{L}$) is an extension of the inner automorphism group Inn($\mathcal{L}$) by a direct product $D_8 \times (0, \infty)$.*

Appendix C

Computation of Continuous Symmetries

The systems corresponding respectively to $\mathbf{v_1},\ \mathbf{v}_2,\ \mathbf{v_6}$ are easy to solve.

$$\begin{cases} \frac{d\hat{t}}{da} &= \frac{1}{\sigma^2} \\ \frac{d\hat{x}}{da} &= \frac{\mathcal{D}}{\sigma^2}\hat{x} \\ \frac{d\hat{u}}{da} &= \frac{r}{\sigma^2}\hat{u} \end{cases} \qquad \begin{cases} \frac{d\hat{t}}{da} &= 0 \\ \frac{d\hat{x}}{da} &= \hat{x} \\ \frac{d\hat{u}}{da} &= 0 \end{cases} \qquad \begin{cases} \frac{d\hat{t}}{da} &= 0 \\ \frac{d\hat{x}}{da} &= 0 \\ \frac{d\hat{u}}{da} &= \hat{u} \end{cases}$$

generate the corresponding continuous symmetries

$$\Gamma_1(a) \begin{cases} \hat{t} &= t + \frac{a}{\sigma^2} \\ \hat{x} &= x \exp\left(\frac{a\mathcal{D}}{\sigma^2}\right) \\ \hat{u} &= u \exp\left(\frac{ar}{\sigma^2}\right) \end{cases} \tag{C.1}$$

$$\Gamma_2(a) \begin{cases} \hat{t} &= t \\ \hat{x} &= x \exp(a) \\ \hat{u} &= u \end{cases} \tag{C.2}$$

$$\Gamma_6(a) \begin{cases} \hat{t} &= t \\ \hat{x} &= x \\ \hat{u} &= u \exp(a) \end{cases} \tag{C.3}$$

In order to simplify the computations, let us assume that x and u (and $\hat{x},\ \hat{u}$ too) are positive variables. It is a natural assumption, giving that x and u represent respectively the price of an asset and the price of a derivative

instrument.
We solve the system associated to $\mathbf{v}_4$.

$$\begin{cases} \frac{d\hat{t}}{da} &= 0 \\ \frac{d\hat{x}}{da} &= \sigma^2 \hat{t}\hat{x} \\ \frac{d\hat{u}}{da} &= (\log \hat{x} - \mathcal{D}\hat{t})\hat{u} \end{cases}$$

from top to bottom. First equation gives $\hat{t} = t$, the second one implies that $\hat{x} = x \exp(a\sigma^2 t)$. Substitute these expressions into the last equation

$$\frac{d\hat{u}}{da} = (\log x - \mathcal{D}t + a\sigma^2 t)\hat{u}$$

rewrite it

$$\frac{d \log \hat{u}}{da} = \log x - \mathcal{D}t + a\sigma^2 t$$

and get the corresponding continuous symmetry

$$\Gamma_4(a) \begin{cases} \hat{t} &= t \\ \hat{x} &= x \exp(a\sigma^2 t) \\ \hat{u} &= u \exp\left[a(\log x - \mathcal{D}t) + \frac{a^2\sigma^2 t}{2}\right] \end{cases} \tag{C.4}$$

Consider the system associated to $\mathbf{v}_3$:

$$\begin{cases} \frac{d\hat{t}}{da} &= 2\hat{t} \\ \frac{d\hat{x}}{da} &= (\log \hat{x} + \mathcal{D}\hat{t})\hat{x} \\ \frac{d\hat{u}}{da} &= \left(2r\hat{t} - \frac{1}{2}\right)\hat{u} \end{cases}$$

First equation gives $\hat{t} = t \exp(2a)$. Last equation could be re-written as follows:

$$\frac{d \log \hat{u}}{da} = 2rt \exp(2a) - \frac{1}{2}$$

and its solution is

$$\hat{u} = u \exp\left\{rt[\exp(2a) - 1] - \frac{a}{2}\right\}$$

In the second equation

$$\frac{d\hat{x}}{da} = [\log \hat{x} + \mathcal{D}t \exp(2a)]\hat{x}$$

we substitute $\log \hat{x} = y$. Using the initial condition $y(0) = \log x$, we get

$$y = \mathcal{D}t[\exp(2a) - \exp(a)] + \log x \exp(a)$$

hence the corresponding continuous symmetry is

$$\Gamma_3(a) \begin{cases} \hat{t} &= t\exp(2a) \\ \hat{x} &= \exp\{\mathcal{D}t[\exp(2a) - \exp(a)] + \log x \exp(a)\} \\ \hat{u} &= u\exp\left\{rt[\exp(2a) - 1] - \frac{a}{2}\right\} \end{cases} \tag{C.5}$$

The system associated to $\mathbf{v}_5$

$$\begin{cases} \frac{d\hat{t}}{da} &= \sigma^2\hat{t}^2 \\ \frac{d\hat{x}}{da} &= \sigma^2\hat{t}\hat{x}\log\hat{x} \\ \frac{d\hat{u}}{da} &= \frac{1}{2}\left[(\log\hat{x} - \mathcal{D}\hat{t})^2 + 2\sigma^2 r\hat{t}^2 - \sigma^2\hat{t}\right]\hat{u} \end{cases}$$

is the most difficult to solve. First equation could be re-written as follows:

$$\frac{d}{da}\left(-\frac{1}{\hat{t}}\right) = \sigma^2$$

Having in mind the initial condition $\hat{t}(0) = t$, we get

$$-\frac{1}{\hat{t}} = \sigma^2 a - \frac{1}{t}$$

which transforms itself into

$$\hat{t} = \frac{t}{1 - a\sigma^2 t}$$

Second equation becomes

$$\frac{d\log\log\hat{x}}{da} = \frac{\sigma^2 t}{1 - a\sigma^2 t}$$

which integrates to

$$\log\log\hat{x} = \log\frac{1}{1 - a\sigma^2 t} + \log\log x$$

and finally gives

$$\hat{x} = \exp\left(\frac{\log x}{1 - a\sigma^2 t}\right)$$

The third equation could be written

$$\frac{d\log\hat{u}}{da} = \frac{1}{2}\left[\frac{(\log x - \mathcal{D}t)^2}{(1-a\sigma^2 t)^2} + \frac{2\sigma^2 r t^2}{(1-a\sigma^2 t)^2} - \frac{\sigma^2 t}{1-a\sigma^2 t}\right]$$

Taking into account that $\log\hat{u}(0) = \log u$, the equation integrates to

$$\log\hat{u} = \frac{1}{2}\left[\frac{(\log x - \mathcal{D}t)^2}{\sigma^2 t(1-a\sigma^2 t)} - \frac{(\log x - \mathcal{D}t)^2}{\sigma^2 t} + \frac{2rt}{1-a\sigma^2 t} - 2rt + \log(1-a\sigma^2 t)\right] + \log u$$

After a couple of simplifications we get the corresponding continuous symmetry

$$\Gamma_5(a)\begin{cases} \hat{t} &= \frac{t}{1-a\sigma^2 t} \\ \hat{x} &= \exp\left(\frac{\log x}{1-a\sigma^2 t}\right) \\ \hat{u} &= u\sqrt{|1-a\sigma^2 t|}\,\exp\left\{\frac{a[(\log x-\mathcal{D}t)^2+2\sigma^2 r t^2]}{2(1-a\sigma^2 t)}\right\} \end{cases} \tag{C.6}$$

Appendix D

Computation of Discrete Symmetries

D.1 Discrete Symmetries of the First Type

Suppose that the discrete symmetry Γ is associated to an automorphism of the type $\varphi_{\varepsilon\varepsilon'}(\delta)$. Its corresponding matrix is $\Theta = \mathrm{diag}(\varepsilon, \varepsilon\varepsilon'\delta, 1, \varepsilon'\delta, \varepsilon, \varepsilon\delta^2)$. We start by solving the first subsystem of equations

$$\begin{cases} \frac{1}{\sigma^2}\frac{\partial \hat{t}}{\partial t} + \frac{\mathcal{D}}{\sigma^2}x\frac{\partial \hat{t}}{\partial x} + \frac{r}{\sigma^2}u\frac{\partial \hat{t}}{\partial u} & = \frac{\varepsilon}{\sigma^2} \\ x\frac{\partial \hat{t}}{\partial x} & = 0 \\ 2t\frac{\partial \hat{t}}{\partial t} + (\log|x| + \mathcal{D}t)x\frac{\partial \hat{t}}{\partial x} + \left(2rt - \frac{1}{2}\right)u\frac{\partial \hat{t}}{\partial u} & = 2\hat{t} \\ \sigma^2 tx\frac{\partial \hat{t}}{\partial x} + (\log|x| - \mathcal{D}t)u\frac{\partial \hat{t}}{\partial u} & = 0 \\ \sigma^2 t^2\frac{\partial \hat{t}}{\partial t} + \sigma^2 tx\log|x|\frac{\partial \hat{t}}{\partial x} + & \\ +\frac{1}{2}[(\log|x| - \mathcal{D}t)^2 + 2\sigma^2 rt^2 - \sigma^2 t]u\frac{\partial \hat{t}}{\partial u} & = \varepsilon\sigma^2\hat{t}^2 \\ u\frac{\partial \hat{t}}{\partial u} & = 0 \end{cases} \tag{D.1}$$

Equations 2 and 6 show that $\hat{t}$ depends only on t. The rest of the subsystem gives

$$\hat{t} = \varepsilon t \tag{D.2}$$

Consider the second subsystem

$$\left\{\begin{array}{lcl}
\frac{1}{\sigma^2}\frac{\partial\hat{x}}{\partial t}+\frac{\mathcal{D}}{\sigma^2}x\frac{\partial\hat{x}}{\partial x}+\frac{r}{\sigma^2}u\frac{\partial\hat{x}}{\partial u} & = & \frac{\varepsilon\mathcal{D}}{\sigma^2}\hat{x}\\
x\frac{\partial\hat{x}}{\partial x} & = & \varepsilon\varepsilon'\delta\hat{x}\\
2t\frac{\partial\hat{x}}{\partial t}+(\log|x|+\mathcal{D}t)x\frac{\partial\hat{x}}{\partial x}+\left(2rt-\frac{1}{2}\right)u\frac{\partial\hat{x}}{\partial u} & = & (\log|\hat{x}|+\mathcal{D}\hat{t})\hat{x}\\
\sigma^2tx\frac{\partial\hat{x}}{\partial x}+(\log|x|-\mathcal{D}t)u\frac{\partial\hat{x}}{\partial u} & = & \varepsilon'\delta\sigma^2\hat{t}\hat{x}\\
\sigma^2t^2\frac{\partial\hat{x}}{\partial t}+\sigma^2tx\log|x|\frac{\partial\hat{x}}{\partial x}+ & & \\
+\frac{1}{2}[(\log|x|-\mathcal{D}t)^2+2\sigma^2rt^2-\sigma^2t]u\frac{\partial\hat{x}}{\partial u} & = & \varepsilon\sigma^2\hat{t}\hat{x}\log|\hat{x}|\\
u\frac{\partial\hat{x}}{\partial u} & = & 0
\end{array}\right. \tag{D.3}$$

Equation 6 shows that $\hat{x}$ does not depend on u. Equations 1, 2 and 4 give

$$\frac{\partial\hat{x}}{\partial t}=\varepsilon\mathcal{D}(1-\varepsilon'\delta)\hat{x}\qquad\frac{\partial\hat{x}}{\partial x}=\varepsilon\varepsilon'\delta\frac{\hat{x}}{x}$$

Equations 3 and 5 reduce themselves to

$$\log|\hat{x}|=\mathcal{D}\varepsilon(1-\varepsilon'\delta)t+\varepsilon\varepsilon'\delta\log|x|$$

that is,

$$|\hat{x}|=\exp[\mathcal{D}\varepsilon(1-\varepsilon'\delta)t+\varepsilon\varepsilon'\delta\log|x|] \tag{D.4}$$

Let us observe that

Remark D.1

$$\log|\hat{x}|-\mathcal{D}\hat{t}=\varepsilon\varepsilon'\delta(\log|x|-\mathcal{D}t)$$

Equation 2 of the third subsystem

$$\left\{\begin{array}{lcl}
\frac{1}{\sigma^2}\frac{\partial\hat{u}}{\partial t}+\frac{\mathcal{D}}{\sigma^2}x\frac{\partial\hat{u}}{\partial x}+\frac{r}{\sigma^2}u\frac{\partial\hat{u}}{\partial u} & = & \frac{\varepsilon r}{\sigma^2}\hat{u}\\
x\frac{\partial\hat{u}}{\partial x} & = & 0\\
2t\frac{\partial\hat{u}}{\partial t}+(\log|x|+\mathcal{D}t)x\frac{\partial\hat{u}}{\partial x}+\left(2rt-\frac{1}{2}\right)u\frac{\partial\hat{u}}{\partial u} & = & \left(2r\hat{t}-\frac{1}{2}\right)\hat{u}\\
\sigma^2tx\frac{\partial\hat{u}}{\partial x}+(\log|x|-\mathcal{D}t)u\frac{\partial\hat{u}}{\partial u} & = & \varepsilon'\delta(\log|\hat{x}|-\mathcal{D}\hat{t})\hat{u}\\
\sigma^2t^2\frac{\partial\hat{u}}{\partial t}+\sigma^2tx\log|x|\frac{\partial\hat{u}}{\partial x}+ & & \\
+\frac{1}{2}[(\log|x|-\mathcal{D}t)^2+2\sigma^2rt^2-\sigma^2t]u\frac{\partial\hat{u}}{\partial u} & = & \frac{\varepsilon}{2}[(\log|\hat{x}|-\mathcal{D}\hat{t})^2+2\sigma^2r\hat{t}^2-\sigma^2\hat{t}]\hat{u}\\
u\frac{\partial\hat{u}}{\partial u} & = & \varepsilon\delta^2\hat{u}
\end{array}\right. \tag{D.5}$$

shows that $\hat{u}$ does not depend on x. Equation 6 gives

$$\frac{\partial\hat{u}}{\partial u}=\varepsilon\delta^2\frac{\hat{u}}{u}$$

and substituting this relationship into the first equation we get that $\hat{u}$ does not depend on t either. Equation 4 is trivially satisfied if we take into account the previous remark. Finally, equations 3 and 5 are solvable if and only if $\varepsilon = \delta = 1$. Hence we are left with the conditions

$$\begin{cases} \frac{\partial \hat{u}}{\partial t} & = & 0 \\ \frac{\partial \hat{u}}{\partial x} & = & 0 \\ \frac{\partial \hat{u}}{\partial u} & = & \frac{\hat{u}}{u} \end{cases}$$

that lead to the solution

$$\hat{u} = \mu u \qquad \mu \neq 0 \tag{D.6}$$

Since we are interested in finding discrete symmetries that are pairwise not equivalent modulo the continuous ones, one may consider $\mu = \pm 1$. The reason for that is the existence of a continuous symmetry $\Gamma_6(\cdot)$ that preserves the independent variables and multiplies the dependent one by an arbitrary positive constant. When $\varepsilon' = 1$, the discrete symmetries we are looking for could be written as follows:

$$\begin{cases} \hat{t} & = & t \\ |\hat{x}| & = & |x| \\ |\hat{u}| & = & |u| \end{cases}$$

while when $\varepsilon' = -1$ we get

$$\begin{cases} \hat{t} & = & t \\ |\hat{x}| & = & \exp(2\mathcal{D}t - \log|x|) \\ |\hat{u}| & = & |u| \end{cases}$$

Dealing with solutions to the standard Black-Scholes equation that represent derivative instruments valuations, we should impose the condition that both prices (of the underlying and of the overlying) be positive. Therefore we can get rid of the absolute values and observe that we have actually found two discrete symmetries, Γ_0 and Γ_*, the first one being the identity and the second one having the following description:

$$\Gamma_* \begin{cases} \hat{t} & = & t \\ \hat{x} & = & \exp(2\mathcal{D}t - \log x) \\ \hat{u} & = & u \end{cases} \tag{D.7}$$

D.2 Discrete Symmetries of the Second Type

Let us apply the same procedure to an automorphism $\psi_{\varepsilon\varepsilon'}(\delta)$. The corresponding matrix is

$$\Theta = \begin{pmatrix} 0 & 0 & 0 & 0 & \varepsilon & 0 \\ 0 & 0 & 0 & \varepsilon'\delta & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & -\varepsilon\varepsilon'\delta & 0 & 0 & 0 & 0 \\ \varepsilon & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \varepsilon\delta^2 \end{pmatrix} \tag{D.8}$$

The first subsystem

$$\left\{ \begin{array}{lcl} \frac{1}{\sigma^2}\frac{\partial \hat{t}}{\partial t} + \frac{\mathcal{D}}{\sigma^2}x\frac{\partial \hat{t}}{\partial x} + \frac{r}{\sigma^2}u\frac{\partial \hat{t}}{\partial u} & = & \varepsilon\sigma^2\hat{t}^2 \\ x\frac{\partial \hat{t}}{\partial x} & = & 0 \\ 2t\frac{\partial \hat{t}}{\partial t} + (\log|x| + \mathcal{D}t)x\frac{\partial \hat{t}}{\partial x} + \left(2rt - \frac{1}{2}\right)u\frac{\partial \hat{t}}{\partial u} & = & -2\hat{t} \\ \sigma^2 tx\frac{\partial \hat{t}}{\partial x} + (\log|x| - \mathcal{D}t)u\frac{\partial \hat{t}}{\partial u} & = & 0 \\ \sigma^2 t^2\frac{\partial \hat{t}}{\partial t} + \sigma^2 tx\log|x|\frac{\partial \hat{t}}{\partial x} + & & \\ +\frac{1}{2}[(\log|x| - \mathcal{D}t)^2 + 2\sigma^2 rt^2 - \sigma^2 t]u\frac{\partial \hat{t}}{\partial u} & = & \frac{\varepsilon}{\sigma^2}\hat{t}^2 \\ u\frac{\partial \hat{t}}{\partial u} & = & 0 \end{array} \right. \tag{D.9}$$

can be solved in the same manner as before. Equations 2, 4 and 6 show that $\hat{t}$ depends only on t, while equations 1, 3 and 5 imply that

$$\hat{t} = -\frac{\varepsilon}{\sigma^4 t} \tag{D.10}$$

The last equation in the second subsystem

$$\left\{ \begin{array}{lcl} \frac{1}{\sigma^2}\frac{\partial \hat{x}}{\partial t} + \frac{\mathcal{D}}{\sigma^2}x\frac{\partial \hat{x}}{\partial x} + \frac{r}{\sigma^2}u\frac{\partial \hat{x}}{\partial u} & = & \varepsilon\sigma^2\hat{t}\hat{x}\log|\hat{x}| \\ x\frac{\partial \hat{x}}{\partial x} & = & \varepsilon'\delta\sigma^2\hat{t}\hat{x} \\ 2t\frac{\partial \hat{x}}{\partial t} + (\log|x| + \mathcal{D}t)x\frac{\partial \hat{x}}{\partial x} + \left(2rt - \frac{1}{2}\right)u\frac{\partial \hat{x}}{\partial u} & = & -(\log|\hat{x}| + \mathcal{D}\hat{t})\hat{x} \\ \sigma^2 tx\frac{\partial \hat{x}}{\partial x} + (\log|x| - \mathcal{D}t)u\frac{\partial \hat{x}}{\partial u} & = & -\varepsilon\varepsilon'\delta\hat{x} \\ \sigma^2 t^2\frac{\partial \hat{x}}{\partial t} + \sigma^2 tx\log|x|\frac{\partial \hat{x}}{\partial x} + & & \\ +\frac{1}{2}[(\log|x| - \mathcal{D}t)^2 + 2\sigma^2 rt^2 - \sigma^2 t]u\frac{\partial \hat{x}}{\partial u} & = & \frac{\mathcal{D}\varepsilon}{\sigma^2}\hat{x} \\ u\frac{\partial \hat{x}}{\partial u} & = & 0 \end{array} \right. \tag{D.11}$$

shows that $\hat{x}$ does not depend on u. Equation 4 gives

$$\frac{\partial \hat{x}}{\partial x} = -\frac{\varepsilon\varepsilon'\delta}{\sigma^2 t}\frac{\hat{x}}{x}$$

Substituting this into equation 5 we get

$$\frac{\partial \hat{x}}{\partial t} = \frac{1}{\sigma^2 t^2}\left(\frac{\mathcal{D}\varepsilon}{\sigma^2} + \varepsilon\varepsilon'\delta \log|x|\right)\hat{x}$$

First equation reduces itself to

$$\log|\hat{x}| = \frac{\varepsilon\varepsilon'\delta\mathcal{D}}{\sigma^2} - \frac{\mathcal{D}\varepsilon}{\sigma^4 t} - \frac{\varepsilon\varepsilon'\delta}{\sigma^2 t}\log|x|$$

which leads to

$$|\hat{x}| = \exp\left[\frac{\varepsilon\varepsilon'\delta\mathcal{D}}{\sigma^2} - \frac{\mathcal{D}\varepsilon}{\sigma^4 t} - \frac{\varepsilon\varepsilon'\delta}{\sigma^2 t}\log|x|\right] \tag{D.12}$$

A simple computation provides the following result.

Remark D.2

$$\log|\hat{x}| - \mathcal{D}\hat{t} = -\frac{\varepsilon\varepsilon'\delta}{\sigma^2 t}(\log|x| - \mathcal{D}t)$$

The last subsystem

$$\left\{\begin{array}{lcl} \frac{1}{\sigma^2}\frac{\partial\hat{u}}{\partial t} + \frac{\mathcal{D}}{\sigma^2}x\frac{\partial\hat{u}}{\partial x} + \frac{r}{\sigma^2}u\frac{\partial\hat{u}}{\partial u} & = & \frac{\varepsilon}{2}[(\log|\hat{x}| - \mathcal{D}\hat{t})^2 + 2\sigma^2 r\hat{t}^2 - \sigma^2\hat{t}]\hat{u} \\ x\frac{\partial\hat{u}}{\partial x} & = & \varepsilon'\delta(\log|\hat{x}| - \mathcal{D}\hat{t})\hat{u} \\ 2t\frac{\partial\hat{u}}{\partial t} + (\log|x| + \mathcal{D}t)x\frac{\partial\hat{u}}{\partial x} + \left(2rt - \frac{1}{2}\right)u\frac{\partial\hat{u}}{\partial u} & = & -\left(2r\hat{t} - \frac{1}{2}\right)\hat{u} \\ \sigma^2 tx\frac{\partial\hat{u}}{\partial x} + (\log|x| - \mathcal{D}t)u\frac{\partial\hat{u}}{\partial u} & = & 0 \\ \sigma^2 t^2\frac{\partial\hat{u}}{\partial t} + \sigma^2 tx\log|x|\frac{\partial\hat{u}}{\partial x} + & & \\ +\frac{1}{2}[(\log|x| - \mathcal{D}t)^2 + 2\sigma^2 rt^2 - \sigma^2 t]u\frac{\partial\hat{u}}{\partial u} & = & \frac{\varepsilon r}{\sigma^2}\hat{u} \\ u\frac{\partial\hat{u}}{\partial u} & = & \varepsilon\delta^2\hat{u} \end{array}\right. \tag{D.13}$$

Equations 2, 4 and 6 give

$$\left\{\begin{array}{lcl} \frac{\partial\hat{u}}{\partial x} & = & -\frac{\varepsilon\delta^2}{\sigma^2 tx}(\log|x| - \mathcal{D}t)\hat{u} \\ \frac{\partial\hat{u}}{\partial u} & = & \varepsilon\delta^2\frac{\hat{u}}{u} \end{array}\right.$$

Substituting these relationships back into the system and using the last remark, we observe that the rest of the equations are compatible if and only if

$$\varepsilon = \delta = 1$$

The system reduces itself to

$$\left\{\begin{array}{lcl} \frac{\partial\hat{u}}{\partial t} & = & \left[-\left(\frac{\mathcal{D}^2}{2\sigma^2} + r\right) + \frac{1}{2t} + \left(\frac{\log^2|x|}{2\sigma^2} + \frac{r}{\sigma^4}\right)\frac{1}{t^2}\right]\hat{u} \\ \frac{\partial\hat{u}}{\partial x} & = & -\frac{1}{\sigma^2 tx}(\log|x| - \mathcal{D}t)\hat{u} \\ \frac{\partial\hat{u}}{\partial u} & = & \frac{\hat{u}}{u} \end{array}\right.$$

Solving this system by the method of characteristics we get

$$\hat{u} = \nu \cdot \sqrt{|t|} \cdot \exp\left\{-\frac{1}{2\sigma^2 t}\left[(\log|x| - \mathcal{D}t)^2 + 2\sigma^2 r t^2 + \frac{2r}{\sigma^2}\right]\right\} \cdot u$$

where ν is a nonzero constant. The same argument that was used before allows us to consider $x, \hat{x}, u, \hat{u} > 0$ and to choose $\nu = \sigma$. Therefore we have found two new discrete symmetries corresponding respectively to $\varepsilon' = 1$ and to $\varepsilon' = -1$, namely

$$\Gamma_+ \begin{cases} \hat{t} &= -\frac{1}{\sigma^4 t} \\ \hat{x} &= \exp\left[-\frac{1}{\sigma^2 t}\left(\log x - \mathcal{D}t + \frac{\mathcal{D}}{\sigma^2}\right)\right] \\ \hat{u} &= \sigma\sqrt{|t|}\exp\left\{-\frac{1}{2\sigma^2 t}\left[(\log x - \mathcal{D}t)^2 + 2\sigma^2 r t^2 + \frac{2r}{\sigma^2}\right]\right\} u \end{cases} \quad \text{(D.14)}$$

$$\Gamma_- \begin{cases} \hat{t} &= -\frac{1}{\sigma^4 t} \\ \hat{x} &= \exp\left[\frac{1}{\sigma^2 t}\left(\log x - \mathcal{D}t - \frac{\mathcal{D}}{\sigma^2}\right)\right] \\ \hat{u} &= \sigma\sqrt{|t|}\exp\left\{-\frac{1}{2\sigma^2 t}\left[(\log x - \mathcal{D}t)^2 + 2\sigma^2 r t^2 + \frac{2r}{\sigma^2}\right]\right\} u \end{cases} \quad \text{(D.15)}$$

D.3 The Discrete Symmetry Group

Let us observe that $\Gamma_+^{-1} = \Gamma_-$ and that $\Gamma_+^2 = \Gamma_-^2 = \Gamma_* = \Gamma_*^{-1}$, which means that the four discrete symmetries listed above form a cyclic group. In fact, we have just proved the following results.

Theorem D.1 *Standard Black-Scholes equation's discrete symmetry group is cyclic of order 4, generated by Γ_+ (or by Γ_-).*

Corollary D.1 *Any symmetry of the standard Black-Scholes equation is a product of the type $\Upsilon\Gamma_+^k$, where Υ is a continuous symmetry and $k \in \{0, 1, 2, 3\}$.*

Appendix E

Properties of the Bivariate Normal Distribution

Theorem E.1 *For any* $a,\ b \in \mathbf{R},\ \rho \in (-1,1)$ *we have*

$$N_2(a,b;\rho) + N_2(a,-b;-\rho) = N(a)$$

$$N_2(a,b;\rho) + N_2(-a,b;-\rho) = N(b)$$

Proof It is enough to prove the first equality. By definition

$$N_2(a,b;\rho)+N_2(a,-b;-\rho) = \int_{-\infty}^{a}\int_{-\infty}^{b} \frac{1}{2\pi}\frac{1}{\sqrt{1-\rho^2}} \exp\left[-\frac{\alpha^2 - 2\rho\alpha\beta + \beta^2}{2(1-\rho^2)}\right] d\beta d\alpha +$$

$$+ \int_{-\infty}^{a}\int_{-\infty}^{-b} \frac{1}{2\pi}\frac{1}{\sqrt{1-\rho^2}} \exp\left[-\frac{\alpha^2 + 2\rho\alpha\beta + \beta^2}{2(1-\rho^2)}\right] d\beta d\alpha$$

After a change of variables $\beta \to -\beta$ we get

$$N_2(a,b;\rho)+N_2(a,-b;-\rho) = \int_{-\infty}^{a}\int_{-\infty}^{b} \frac{1}{2\pi}\frac{1}{\sqrt{1-\rho^2}} \exp\left[-\frac{\alpha^2 - 2\rho\alpha\beta + \beta^2}{2(1-\rho^2)}\right] d\beta d\alpha +$$

$$+ \int_{-\infty}^{a}\int_{b}^{\infty} \frac{1}{2\pi}\frac{1}{\sqrt{1-\rho^2}} \exp\left[-\frac{\alpha^2 - 2\rho\alpha\beta + \beta^2}{2(1-\rho^2)}\right] d\beta d\alpha =$$

$$= \int_{-\infty}^{a}\int_{-\infty}^{\infty} \frac{1}{2\pi}\frac{1}{\sqrt{1-\rho^2}} \exp\left[-\frac{\alpha^2 - 2\rho\alpha\beta + \beta^2}{2(1-\rho^2)}\right] d\beta d\alpha =$$

$$= \int_{-\infty}^{a}\int_{-\infty}^{\infty} \frac{1}{2\pi}\frac{1}{\sqrt{1-\rho^2}} \exp\left[-\frac{(1-\rho^2)\alpha^2 + (\beta-\rho\alpha)^2}{2(1-\rho^2)}\right] d\beta d\alpha =$$

$$= \int_{-\infty}^{a} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{\alpha^2}{2}\right) \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{\gamma^2}{2}\right) d\gamma d\alpha =$$

$$= \int_{-\infty}^{a} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{\alpha^2}{2}\right) d\alpha = N(a)$$

where

$$\gamma = \frac{\beta - \rho\alpha}{\sqrt{1-\rho^2}}$$

Corollary E.1 *For any* $a,\ b \in \mathbf{R},\ \rho \in (-1,1)$ *we have*

$$N_2(a,b;\rho) - N_2(-a,-b;\rho) = N(b) - N(-a)$$

Proof According to Theorem E.1, we may write

$$N_2(a,b;\rho) - N_2(-a,-b;\rho) = N_2(a,b;\rho) - N(-a) + N_2(-a,b;-\rho) =$$

$$= N_2(a,b;\rho) - N(-a) + N(b) - N_2(a,b;\rho) = N(b) - N(-a)$$

Appendix F

A Remarkable Symmetry

Theorem F.1 *Let $u(\cdot,\cdot)$ be a derivative valuation. Then*

$$\tilde{\Gamma}[u(t,x)] := \left(\frac{H}{x}\right)^{\frac{2\mathcal{D}}{\sigma^2}} u\left(t, \frac{H^2}{x}\right) \qquad \forall t, x$$

is another derivative valuation for any positive constant H.

Proof It is enough to observe that $\tilde{\Gamma}$ is a product of (SBS) equation's symmetries. To be more precise, let us check that

$$\tilde{\Gamma} = \Gamma_6(a'') \circ \Gamma_2(a') \circ \Gamma_4(a) \circ \Gamma_*$$

where

$$a = -\frac{2\mathcal{D}}{\sigma^2} \qquad a' = 2\log H \qquad a'' = -\frac{2\mathcal{D}}{\sigma^2}\log H$$

The successive actions of the operators $\Gamma_*,\ \Gamma_4(a),\ \Gamma_2(a'),\ \Gamma_6(a'')$ on any $t,\ x,\ u$ can be described as follows.

$$\left\{\begin{array}{l} t \\ x \\ u \end{array}\right. \mapsto \left\{\begin{array}{l} t \\ \exp(2\mathcal{D}t - \log x) \\ u \end{array}\right. \mapsto \left\{\begin{array}{l} t \\ \exp(-\log x) \\ u\exp\left(\frac{2\mathcal{D}}{\sigma^2}\log x\right) \end{array}\right. \mapsto$$

$$\mapsto \left\{\begin{array}{l} t \\ \exp(2\log H - \log x) \\ u\exp\left(\frac{2\mathcal{D}}{\sigma^2}\log x\right) \end{array}\right. \mapsto \left\{\begin{array}{l} t \\ \frac{H^2}{x} \\ \left(\frac{x}{H}\right)^{\frac{2\mathcal{D}}{\sigma^2}} u \end{array}\right.$$

Denote $(\hat{t}, \hat{x}, \hat{u}) = \tilde{\Gamma}(t, x, u)$. Then

$$\tilde{\Gamma} \begin{cases} \hat{t} = t \\ \hat{x} = \frac{H^2}{x} \\ \hat{u} = \left(\frac{x}{H}\right)^{\frac{2\mathcal{D}}{\sigma^2}} u \end{cases} \qquad \tilde{\Gamma}^{-1} \begin{cases} t = \hat{t} \\ x = \frac{H^2}{\hat{x}} \\ u = \left(\frac{x}{H}\right)^{-\frac{2\mathcal{D}}{\sigma^2}} \hat{u} \end{cases} \equiv \begin{cases} t = \hat{t} \\ x = \frac{H^2}{\hat{x}} \\ u = \left(\frac{H}{\hat{x}}\right)^{-\frac{2\mathcal{D}}{\sigma^2}} \hat{u} \end{cases}$$

Hence the transformed solution $\hat{u}(\cdot, \cdot)$ can be expressed as

$$\hat{u}(\hat{t}, \hat{x}) = \left(\frac{x}{H}\right)^{\frac{2\mathcal{D}}{\sigma^2}} u(t, x) = \left(\frac{H^2/\hat{x}}{H}\right)^{\frac{2\mathcal{D}}{\sigma^2}} u\left(\hat{t}, \frac{H^2}{\hat{x}}\right) = \left(\frac{H}{\hat{x}}\right)^{\frac{2\mathcal{D}}{\sigma^2}} u\left(\hat{t}, \frac{H^2}{\hat{x}}\right)$$